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Mini Review

A remark on a perturbed Benjamin-Bona-Mahony type equation and its complete integrability

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Abstract

In the Letter, we study a perturbed Benjamin-Bona-Mahony nonlinear equation, which was derived for describing shallow water waves and possessing a rich Lie symmetry structure. Based on the gradient-holonomic integrability checking scheme applied to this equation, we have analytically constructed its infinite hierarchy of conservation laws, derived two compatible Poisson structure and stated its complete integrability.

Introduction

The well-known Benjamin-Bona-Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.1}$$

Where $u \in C^\infty(\mathbb{R};\mathbb{R})$ is a smooth real-valued function, symbols "$u_{xt...}$" denote its partial derivatives with respect to spatial $x \in \mathbb{R}$ and temporal $t \in \mathbb{R}$ parameters, describe nonlinear waves on shallow water and were derived for the first time in [1] and reintroduced later in [2]. It was recently mentioned [3] that, as this equation lacks a Galilean symmetry and loses some important properties, this equation (1.1) was suitably perturbed with some higher order terms that helped to recover this symmetry, yet this remedy appeared to lead to the simultaneous loss of the presence of the energy conservation law. A more detailed analysis of this aspect led the authors of the recent work [4] to study the following perturbed version of the equation (1.1):

$$u_t - u_{xxt} = -u_x - uu_x + \alpha(uu_{xxx} + 2u_x u_{xx}) \tag{1.2}$$

For different real parameters $\alpha \in \mathbb{R}$, where they stated by means of standard computer-assisted Lie symmetry analysis [5] that at the value $\alpha = 1/3$ the evolution flow (1.2) possesses a wide hierarchy of Lie symmetries and additionally a one Lie-Backlund [5] symmetry

$$u_\tau = \frac{u_x - u_{xxx}}{[2(u - u_{xx}) + 3]^{3/2}} \tag{1.3}$$

With respect to an evolution parameter $\tau \in \mathbb{R}$. Moreover, based on analysis of the calculated symmetries, the authors of the work [4] show that the flow (1.2) is Hamiltonian:

$$u_t = -\vartheta\, grad \tilde{H}_1[u], \quad \tilde{H}_1 = \frac{1}{2}\int [u^2 + 1/3(u^3 + uu_x^2)]dx, \tag{1.4}$$

With respect to the Poisson operator

$$\vartheta := (\partial^{-1} - \partial)^{-1}, \tag{1.5}$$

Where, by definition, $\partial := \partial/\partial x, x \in \mathbb{R}$, and put forward a hypothesis that the evolution flow (1.2) is closely related to the well known Camassa-Holm equation [6,7] and is a completely integrable [8-10] Hamiltonian system.

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Our Letter is devoted to strong analytical proving this hypothesis both for the Hamiltonian system (1.4) and its symmetry flow (1.3).

Integrability analysis

First we observe that the perturbed nonlinear BBM [2] type equation (1.2) being equivalent to the nonlinear dynamical system

$$u_t = -(1-\partial_x^2)^{-1}[u_x + uu_x - 1/3(uu_{xxx} + 2u_x u_{xx}) := \tilde{F}[u] \tag{2.1}$$

On a functional manifold $M \subset C^\infty(\mathbb{R};\mathbb{R})$ of smooth real valued functions, possesses as its symmetry the evolution flow

$$u_\tau = \frac{u_x - u_{3x}}{[2(u-u_{xx})+3]^{3/2}} := \tilde{K}[u], \tag{2.2}$$

Which jointly with (2.1) are commuting to each other, that is

$$\left[\tilde{K}, \tilde{F}\right] = 0 \tag{2.3}$$

For all smooth functions $u \in M$. Moreover, it is easy to check that the flows (2.1) and (2.2) above reduce via the argument transformation $u(x,t) \to v(x-1/2t,t) - 3/2$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}$ to the following equivalent Hamiltonian forms:

$$v_t = -(\partial^{-1} - \partial)^{-1} grad H_1[v] := -\vartheta grad H_1[v] := F[v], \quad H_1 := \frac{1}{2}\int (v^2 + v_x^2)(v/3-1)dx, \tag{2.4}$$

With respect to the evolution parameter $t \in \mathbb{R}$ and

$$v_\tau = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{v - v_{xx}}}\right) = -\vartheta grad H_2[v] := K[v], \quad H_2 = 2\int\sqrt{v - v_{xx}}\,dx, \tag{2.5}$$

With respect to the evolution parameter $\tau \in \mathbb{R}$, respectively, where the skew-symmetric Poisson operator $\vartheta: T^*(M) \to T(M)$ naturally acts as a pseudo-differential expression [11,12] from the cotangent space $T^*(M)$ to the tangent space $T(M)$ over the functional Poisson manifold M and determines the related Poisson bracket

$$\{\cdot,\cdot\} := (grad(\cdot) \,|\, \vartheta grad(\cdot)) \tag{2.6}$$

Over the space of functionals on M endowed with the natural bilinear form $(\cdot|\cdot): T^*(M)\times T(M) \to \mathbb{R}$.

Being interested in proving the suspected complete integrability of these flows, we take into account that the flows (2.4) and (2.5) commute to each other, thus yielding that it is enough to check the integrability of the second looking more simply flow (2.5) on a smooth functional manifold M and show that the first flow (2.1) enters into a innfinite hierarchy of commuting to each other Hamiltonian flows generated by the second flow (2.5). Within the nowadays existing three effective enough integrability checking schemes [13-15], we stopped on the gradient-holonomic integrability checking scheme [14], within which one needs first [16,17] to find an asymptotic solution to a Baker-Akhiezer type function $\varphi(x;\lambda) \in T^*(M)$, $\lambda \in \Gamma$, defined [8,9,14,18] on the spectrum Γ of a suitably chosen Lax type [8,10,18] operator, equivalent to the recursion operator Λ: $T^*(M) \to T^*(M)$, being its eigenfunction with some iso-spectral eigenvalue, whose value does not matter for us. This, in particular, means that the following functional-operator equations on the manifold M are compatible:

$$\Lambda\varphi(x;\lambda) = \xi(\lambda)\varphi(x;\lambda) \tag{2.7}$$

and

$$d\Lambda / dt = [\Lambda, K^{',*}], \tag{2.8}$$

Where the latter follows from the determining Lax-Noether linear equation

$$\varphi_t + K^{',*}\varphi = 0 \tag{2.9}$$

And the Magri type [14,15,19] symmetry hereditary property, whose asymptotic $\lambda \to \infty \in \Gamma$ solution is in general representable as

$$\text{è} \sim \exp[-\lambda^3 t + \partial^{-1}(\sigma_{-1}\lambda^1 + \sigma_0 + \sigma_1\lambda^{-1} + \sigma_2\lambda^{-2} + ...)], \tag{2.10}$$

Where the functionals

$$\gamma_j := \int \sigma_j[v]dx \tag{2.11}$$

Are, by construction, the corresponding conservation laws of the Hamiltonian system (2.5), that is

$$\frac{d}{dt}\gamma_j\Big|_{v_t = K[v]} = 0 \tag{2.12}$$

For all $j \in \mathbb{Z}_+ \cup \{-1\}$. In particular, one easily obtains recurrently that

$$\sigma_{-1} = 2^{\frac{1}{3}}\sqrt{v - v_{xx}}, \sigma_0 = \frac{1}{2}\frac{d}{dx}\ln(v_{xx} - v), \tag{2.13}$$

$$\sigma_1 = \frac{8(v - v_{xx})v_{xxxx} + 13v_{xxx}^2 - 26v_x v_{xxx} + 12v_{xx}^2 - 16vv_{xx} + 13v_x^2 + 4v^2}{3\cdot 2^{\frac{10}{3}}(v - v_{xx})^{\frac{5}{2}}}$$

Whence conservation laws equal

$$\gamma_{-1} = 2^{1/3}\int\sqrt{v - v_{xx}}\,dx = 2^{1/3}H_2, \gamma_0 = \frac{1}{2}\int\frac{d}{dx}\ln(v_{xx} - v)dx = 0, \tag{2.14}$$

$$\gamma_1 = \int dx\frac{8(v - v_{xx})v_{xxxx} + 13v_{xxx}^2 - 26v_x v_{xxx} + 12v_{xx}^2 - 16vv_{xx} + 13v_x^2 + 4v^2}{3\cdot 2^{\frac{10}{3}}(v - v_{xx})^{\frac{5}{2}}}, ...,$$

And so on. Moreover, as these conservation laws naturally generate an infinite hierarchy of commuting to each other Hamiltonian flows

$$v_{t_j} := -(\partial_x^{-1} - \partial_x)^{-1} grad\gamma_j[v] := K_j[v], \quad [K_j, K_s] = 0, \tag{2.15}$$

as well as

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$$\{\gamma_j, \gamma_s\} = 0 \tag{2.16}$$

For all $j, s \in \mathbb{Z}_+ \cup \{-1\}$ with respect to the related evolution parameters $t_j \in \mathbb{R}, j \in \mathbb{Z}_+ \cup \{-1\}$, we can deduce that our dynamical system (2.5) and respectively, its commuting symmetry - the system (2.4), are jointly integrable Hamiltonian flows on the functional manifold M. Taking into account the resulting existence of the recursion operator $\wedge : T^*(M) \to T^*(M)$, satisfying the second equation of (2.5) and related gradient recursion relationships

$$\Lambda grad\gamma_j = grad\gamma_{j+2} \tag{2.17}$$

For all, $j \in \mathbb{Z}_+ \cup \{-1\}$, we also naturally derive the existence of the second Poisson operator

$$\eta := \vartheta\Lambda : T^*(M) \to T(M), \tag{2.18}$$

Compatible [10,17,18] with the first Poisson operator $\vartheta : T^*(M) \to T(M)$ on the functional manifold M, that is the sum $(\vartheta + \lambda\eta) : T^*(M) \to T(M)$ persists to be Poissonian for all $\lambda \in \mathbb{R}$. The exact expression for the second Poisson operator is easily obtained from the operator relationship, naturally following from the Lax-Noether equation (2.9) in the case when its solution $\psi \in T^*(M)$ is not symmetric, that is $\psi' \neq \psi'^{,*}$ on the whole manifold M. It is easy to check that the conservation law $H_1 = \frac{1}{2}\int (v^2 + v_x^2)(v/3 - 1)dx$ can be represented in the following equivalent form

$$H_1 = (1/3v_x v - 1/3\partial^{-1}v^2 \,|\, v_x) + (\partial^{-1}v - v_x \,|\, v_x) := (\psi_1 \,|\, v_x) \tag{2.19}$$

With respect to the standard bilinear form $(\cdot|\cdot) : T^*(M) \times T(M) \to \mathbb{R}$ on the product $T^*(M) \times T(M)$, where $\psi_1 = 1/3v_x v - 1/3\partial^{-1}v^2 + \partial^{-1}v - v_x \in T^*(M)$ does satisfy the Lax-Noether equation (2.9): $\psi_t + K'^{,*}\psi = 0\,(mod grad L)$ for some smooth function L: $M \to R$ on the whole manifold M Then, it is easy to observe that the operator

$$\begin{aligned}\eta_1^{-1} = \psi_1' - \psi_1'^{,*} &= 2\left(\partial^{-1} - \partial\right) + 1/3(\partial \circ v + v\partial) - 2/3(v\partial^{-1} + \partial^{-1}v) = \\ &= 2\vartheta^{-1} + 2/3(\sqrt{v}\partial\sqrt{v} - 2\partial^{-1}\sqrt{v}\partial\sqrt{v}\partial^{-1}).\end{aligned} \tag{2.20}$$

Is a *priori* symplectic on the functional manifold M jointly with the expression

$$\eta^{-1} = \sqrt{v}\partial\sqrt{v} - 2\partial^{-1}\sqrt{v}\partial\sqrt{v}\partial^{-1}, \tag{2.21}$$

Since a linear combination of symplectic operators, if nondegenerate, is always [20,21] symplectic. The usual way one check that the operators ϑ, $\eta : T^*(M) \to T(M)$ are, as expected, compatible with M, that is the operator $\eta^{-1}\vartheta$, $\eta^{-1} : T(M) \to T^*(M)$ proves to be also symplectic. Similarly one can find that

$$H_2 = 2\int\sqrt{v - v_{xx}}\,dx := (\psi_2 \,|\, v_x) = \left((\partial - \partial^{-1})\left(\frac{2}{\sqrt{v - v_{xx}}}\right) \middle| v_x \right), \tag{2.22}$$

Providing the generating element

$\psi_2 = (\partial - \partial^{-1})\left(\frac{2}{\sqrt{v - v_{xx}}}\right) \in T^*(M)$, which gives rise to the second compatible symplectic operator

$$\eta_2^{-1} = \psi_2' - \psi_2'^{,*} = (\partial^{-1} - \partial)\left(\frac{1}{(v - v_{xx})^{3/2}}\partial + \partial\frac{1}{(v - v_{xx})^{3/2}}\right)(\partial^{-1} - \partial). \tag{2.23}$$

Moreover, as there holds the representation $\eta_2^{-1} = \vartheta^{-1}\eta_3\vartheta^{-1}$, we easily derive that the operator expression

$$\eta_3 = \frac{1}{(v - v_{xx})^{3/2}}\partial + \partial\frac{1}{(v - v_{xx})^{3/2}} \tag{2.24}$$

Presents the third compatible Poisson operator $\eta : T^*(M) \to T(M)$ for the Hamiltonian system (2.5) on the functional manifold M and determines the recursion operator

$$\Lambda = \eta^{-1}\vartheta = \sqrt{v}\partial\sqrt{v} - 2\partial^{-1}\sqrt{v}\partial\sqrt{v}\partial^{-1})(\partial^{-1} - \partial)^{-1}, \tag{2.25}$$

Satisfying the relationships (2.17), which can be naturally extended back to the negative values of indices $j \in \mathbb{Z}$:

$$\Lambda grad\gamma_j = grad\gamma_{j+2.} \tag{2.26}$$

In particular, one can calculate that $\Lambda^{-1}grad\gamma_{-1} = grad\gamma_{-3}$, where $\gamma_{-3} = H_1$, yielding right away that the Hamiltonian flow (2.4) possesses an infinite hierarchy of commuting to each other conservation laws (2.14), thus being a completely integrable Hamiltonian system on the manifold M.

Thus, we have stated the following summarizing integrability theorem.

Theorem 2.1 *The nonlinear dynamical systems (2.4) and (2.5) possess a common infinite hierarchy of commuting to each other conservation laws (2.14) with respect to both Poisson brackets (1.5) and (2.21), thus presenting on the smooth functional Poisson manifold M related to each other completely integrable bi-Hamiltonian systems.*

As a natural conclusion from the presented above integrability analysis of the perturbed nonlinear BBM type equation (1.2), one can state the true analytical effectiveness of the gradient-holonomic integrability checking scheme. Its implementing too many other nonlinear dynamical systems, possessing a rich Lie symmetry structure, and modeling diverse evolution phenomena in physical and biological sciences, is expected to be both mathematically attractive and useful.

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