Peertechz

MATHEMATICS AND PHYSICS GROUP

Annals of Mathematics and Physics — Open Access Journal

ISSN: 2689-7636 DOI: https://dx.doi.org/10.17352/amp

Review Article

Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement

Abraham A Ungar*

Department of Mathematics, North Dakota State University, USA

Received: 14 May, 2024
Accepted: 31 May, 2024
Published: 03 June, 2024

***Corresponding author:** Abraham A Ungar, Department of Mathematics, North Dakota State University, USA, Email: abraham.ungar@ndsu.edu, ungarav@gmail.com

ORCiD: https://orcid.org/0000-0003-2882-1663

Keywords: Special relativity; Lorentz transformation of signature (m,n); Galilei transformation of signature (m,n); Multi-particle entanglement

https://www.mathematicsgroup.us

Abstract

Special relativity theory stems from the Lorentz transformation of signature (1,3). The incorporation into special relativity of the Lorentz transformations of signature (m,n) for all $m,n \in N$ (n = 3 in physical applications) enriches the theory. The resulting enriched special relativity is a friendly extended special relativity that admits multi-particle entanglement, as demanded by relativistic quantum mechanics. The Lorentz transformation of signature (m,n) admits a novel physical interpretation induced by the intuitively clear interpretation of the Galilei transformation of signature (m,n) for all $m,n > 1$. In this sense we understand Lorentz utilizing Galilei in m temporal and n spatial dimensions, resulting in the emergence of multi-particle entanglement that the enriched special theory of relativity admits. Remarkably, it turns out that, for any $m,n \in N$, the group of Lorentz transformations of signature (m,n) is the symmetry group that underlies any multi-particle system that consists of m n - dimensional entangled particles.

Introduction

Nature organizes itself using the language of symmetries. Thus, in particular, the underlying symmetry group by which Einstein's special relativity theory can be understood is the Lorentz group $SO_c(1,3)$ of Lorentz transformations of signature (1,3). A physical system obeys the Lorentz symmetry if the relevant laws of physics are invariant under Lorentz transformations. Lorentz symmetry is one of the cornerstones of modern physics. However, it is known that entanglement in quantum mechanics involves Lorentz symmetry violation [1-6]. Indeed, several explorers exploit entangled particles to observe Lorentz symmetry violation; see, for instance [7-13].

Quantum entanglement [14] was named by Einstein as ``spooky action at a distance''. It is a physical phenomenon that occurs when groups of particles interact in ways such that the quantum state of each particle cannot be described independently of the others, *even when the particles are separated by a large distance*. Instead, a quantum state must be described as a system of particles as a whole.

Understanding entanglement in relativistic settings has been a key question in relativistic quantum mechanics. Some results show that entanglement is observer-dependent [2]. The aim of this review article is, therefore, to present results that demonstrate in extended relativistic settings the following: For any $m,n \in N$, the Lorentz transformation group $SO_c(m,n)$ of Lorentz transformations of signature (m,n) is the missing symmetry group that underlies any multi-particle system of m entangled n - dimensional particles.

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118

A Lorentz boost is a Lorentz transformation without rotations. The Lorentz boost $B_c(V)$ of signature (m,n), $m,n\in N$, in m temporal and n spatial dimensions is parametrized by an $n \times m$ velocity matrix V. It is introduced in Section 2, giving rise to the intuitively clear Galilei boost $B_\infty(V)$ of signature (m,n) in Section 5. The Lorentz boost $B_c(V)$ and its associated Galilei boost $B_\infty(V)$ are related by a novel additive decomposition $B_c(V)$ into the sum of two components: (i) a Galilean component $B_\infty(V)$ and (ii) an entanglement component $c^{-2}E(V)$. The additive decomposition is presented in (16) for $m = 1$ and $n\in N$, paving the road to the presentation of the additive decomposition in (18) for all $m,n\in N$.

The additive decomposition enables the counterintuitive Lorentz boost of signature (m,n) to be understood utilizing the intuitively clear Galilei boost of signature (m,n). The concept of understanding Lorentz utilizing Galilei, thus, stems from the additive decomposition in (16) and (18).

The resulting idea of understanding Lorentz utilizing Galilei suggests that the group $SO_c(m,n)$ of all Lorentz transformations of signature (m,n) is the symmetry group that underlies any multi-particle system of m n dimensional entangled particles, for $m,n\in N$. This interpretation, according to which the Lorentz group of signature (m,n) $m>1$ is the symmetry group of a multi-particle system of m n --dimensional particles, is based on mathematical structures and analogies with experimentally supported results. Hopefully, therefore, this article will stimulate a search for experimental support for our physical interpretation of the Lorentz group $SO_c(m,3)$ of signatures $(m,3)$, $m>1$. Finally, a search for experimental support that involves the shifting of energy levels that, according to [10], results from quantum entanglement is suggested.

Lorentz boost of signature (m,n): Parametric realization

Let $\mathbb{R}^{m,n}$ be a pseudo-Euclidean space of signature (m,n) of m temporal dimensions and n spatial dimensions, $m,n\in N$. A linear transformation Λ $\mathbb{R}^{m,n}$ is a Lorentz transformation of signature(m,n), or (m,n)-Lorentz transformation, if it leaves the squared pseudo norm

$$\sum_{i=1}^{m} t_i^2 - \frac{1}{c^2}\sum_{i=1}^{n} x_i^2 \qquad (1)$$

invariant and can be reached continuously from the identity transformation $\mathbb{R}^{m,n}$. Here c is an arbitrarily fixed positive constant that represents the vacuum speed of light. The group of all (m,n)-Lorentz transformations is denoted by $SO_c(m,n)$. A Lorentz boost of signature (m,n), or (m,n)-Lorentz boost, is an (m,n)-Lorentz transformation without rotations. With $c = 1$, the (m,n)-Lorentz transformations are known as proper pseudo-orthogonal transformations [15, p.~478].

A novel, unified parametric realization of the set of all (m,n)-Lorentz boosts for any $m,n\in N$ is discovered in [16], obtaining the elegant $(m + n) \times (m + n)$ parametric matrix representation in (7) as follows:

Let $\mathbb{R}^{n\times m}$ be the set of all real $n\times m$ matrices and let $\mathbb{R}_c^{n\times m} \subset \mathbb{R}^{n\times m}$ be the c-ball of $\mathbb{R}^{n\times m}$ given by

$$\mathbb{R}_c^{n\times m} = \{V \in \mathbb{R}^{n\times m} : \|V\| < c\} \qquad (2)$$

Where $||V||$ is the matrix spectral norm of V [17, p.~295], [16, Definition 5.7]. It should be noted here that in the special case when $m = 1$ the matrix $V \in\mathbb{R}^{n\times 1}$ can be viewed as a vector and, as such, the matrix spectral norm of the matrix $V \in\mathbb{R}^{n\times 1}$ and its Euclidean norm coincide.

Let $V \in\mathbb{R}^{n\times m}$. Each of the two real symmetric matrices

$$\begin{aligned} \Gamma_{n,V,c}^{L} &:= \sqrt{I_n - c^{-2}VV^t}^{\,-1} \in \mathbb{R}^{n\times n} \\ \Gamma_{m,V,c}^{R} &:= \sqrt{I_m - c^{-2}V^tV}^{\,-1} \in \mathbb{R}^{m\times m} \end{aligned} \qquad (3)$$

exists if and only if $V \in \mathbb{R}_c^{n\times m}$ [16, Section 5.3]. Here, I_n is the $n\times n$ identity matrix, and exponent t denotes transposition.

It is convenient to use the short notation $\Gamma_V^L = \Gamma_{n,V,c}^L$ and $\Gamma_V^R = \Gamma_{m,V,c}^R$, noting that c is an arbitrarily fixed positive constant and that the signature parameters (m,n) are recovered from the dimensions of the matrix parameter $V \in\mathbb{R}^{n\times m}$.

In the special case when $m = 1$, $V \in \mathbb{R}_c^{n\times 1}$ is a column vector, $V^tV = V^2 < c^2$, and

$$\Gamma_V^R = \sqrt{1 - c^{-2}V^2}^{\,-1} = \gamma_V \in \mathbb{R}^{1\times 1} = \mathbb{R} \quad (m=1) \qquad (4)$$

is the Lorentz gamma factor of special relativity. Accordingly, Γ_V^R is called the right gamma factor of signature (m,n) and Γ_V^L is called the left gamma factor of the signature (m,n). Hence, the Lorentz gamma factor γ_V is the right gamma factor of signature $(1,n)$.

Remark *(Matrix Division Notation)*. Let M_1 and M_2 be two matrices such that the inverse, M_2^{-1}, of M_2 exists. If the two matrices satisfy the commuting relation

$$M_1M_2^{-1} = M_2^{-1}M_1,$$

then we may adopt the *matrix division notation*

$$\frac{M_1}{M_2} := M_1M_2^{-1} = M_2^{-1}M_1.$$

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118

It is important to note that the left and right gamma factors of any signature (m,n), $m,n \in N$, obey the identities [16, Lemma 5.82]

$$\begin{aligned} \Gamma_V^L &= I_n + \frac{1}{c^2}\frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L} V V^t \\ \Gamma_V^R &= I_m + \frac{1}{c^2}\frac{(\Gamma_V^R)^2}{I_m+\Gamma_V^R} V^t V \end{aligned} \tag{5}$$

for all $V \in \mathbb{R}_c^{n\times m}$. Identities (5) will prove useful in constructing the additive decomposition (18).

In the special case when $m = 1$ the second identity in (5) descends to the identity that γ_V obeys,

$$\gamma_V = 1 + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V} V^2 \quad (m=1) \tag{6}$$

for all $V \in \mathbb{R}_c^n$. Identity (6) will prove useful in constructing the additive decomposition (16).

Finally, as shown in [16, Eq.~5.128], for any $m,n \in N$, the set of all (m,n)-Lorentz boosts has the elegant $(m+n)\times(m+n)$ block matrix representation

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \in SO_c(m,n), \tag{7}$$

parametrized by the matrix $V \in \mathbb{R}_c^{n\times m}$. The physical interpretation of the matrix parameter $V \in \mathbb{R}_c^{n\times m}$ will be revealed in Section 5 utilizing the intuitively clear (m,n)-Galilei boost that we introduce in that section.

Beautifully, the submatrices of $B_c(V)$ in (7) illustrate the symmetry between m-dimensional time and n-dimensional space in terms of the right and left gamma factors. On this occasion, it is interesting to note that the right and left gamma factors enjoy the *commuting relations* [16, Eq.~(5.119)]

$$\begin{aligned} \Gamma_V^L V &= V \Gamma_V^R \\ \Gamma_V^R V^t &= V^t \Gamma_V^L \\ \Gamma_V^L V V^t &= V V^t \Gamma_V^L \\ \Gamma_V^R V^t V &= V^t V \Gamma_V^R. \end{aligned} \tag{8}$$

For any signature (m,n), the matrix $B_c(V)$, where $V \in \mathbb{R}_c^{n\times m}$, is a linear transformation that takes a time-space event

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{x} \end{pmatrix} := (t_1,\ldots,t_m,x_1,\ldots,x_n)^t \in \mathbb{R}^{m,n} \tag{9}$$

to a time-space event

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{x}' \end{pmatrix} := (t_1',\ldots,t_m',x_1',\ldots,x_n')^t \in \mathbb{R}^{m,n} \tag{10}$$

leaving the squared pseudo norm invariant,

$$\mathbf{t}^2 - \frac{1}{c^2}\mathbf{x}^2 = (\mathbf{t}')^2 - \frac{1}{c^2}(\mathbf{x}')^2, \tag{11}$$

where $\mathbf{t},\mathbf{t}' \in \mathbb{R}^m$ and $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$.

An elegant, straightforward proof of the result in (11), based on the commuting relations in (8), is found in [16, Sect.~5.8]. The unified parametrization of the Lorentz (m,n)-boost in (7) for any $m,n \in N$, thus, shakes down the underlying matrix algebra into elegant and transparent results.

The special case when $m = 1$

The special case when $m = 1$ is obvious. In this section, we present this special case to pave the road to the general case when $m \in N$ there is any natural number.

In the special case when $m = 1$ the (m,n)-Lorentz boost (7) descends to the standard $(1,n)$ -Lorentz boost of special relativity (where $n = 3$ in physical applications), which is

$$B_c(V) = \begin{pmatrix} \gamma_V & \frac{1}{c^2}\gamma_V V^t \\ \gamma_V V & I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V} V V^t \end{pmatrix} \in SO_c(1,n) \tag{12}$$

where $V \in \mathbb{R}_c^{n\times 1} \subset \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector in the ball $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ of the Euclidean n-space $\mathbb{R}^n$, $\mathbb{R}_c^n = \{V \in \mathbb{R}^n : \|V\| < c\}$, and where γ_v is the Lorentz gamma factor given by (4). The proof that (7) descends to (12) in the special case when $m = 1$ is presented in [16, Eq.~5.173].

In the Galilean limit, $c \to \infty$, the $(1,n)$ -Lorentz boost $B_c(V)$ in (12) tends to the $(1,n)$ -Galilei boost $B_\infty(V)$,

$$\lim_{c\to\infty} B_c(V) = B_\infty(V) = \begin{pmatrix} 1 & 0_{1\times n} \\ V & I_n \end{pmatrix} \tag{13}$$

where $0_{m\times n}$ is the $m\times n$ zero matrix.

Contrasting the $(1,n)$-Lorentz boost $B_c(V)$, the $(1,n)$ -Galilei boost $G_\infty(V)$ is intuitively clear. Thus, for instance, for $n = 3$ and

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118

$$V = \mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^{3\times 1} \tag{14}$$

we have the following boost application:

$$B_\infty(V)\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix}. \tag{15}$$

Equation (15) indicates the physical interpretation of the (1,3)-Galilei boost $B_\infty(V)$, according to which it boosts by velocity $V = \mathbf{v} = (v_1, v_2, v_3)^t$ a single particle in a position $\mathbf{x} = (x_1, x_2, x_3)^t$ at the time t to the boosted position $\mathbf{x} + \mathbf{v}t$.

Owing to Identity (6), the (1,n)-Lorentz boost $B_c(V)$ in (12) possesses the remarkable *additive decomposition* as the sum of a Galilean component $B_\infty(V)$ and an entanglement component $c^{-2}E(V)$ given by

$$B_c(V) = \begin{pmatrix} 1 & 0_{1\times n} \\ V & I_n \end{pmatrix} + \frac{1}{c^2}\gamma_V \begin{pmatrix} \frac{\gamma_V}{1+\gamma_V}V^2 & V^t \\ \frac{\gamma_V}{1+\gamma_V}V^2 V & \frac{\gamma_V}{1+\gamma_V}VV^t \end{pmatrix}$$

$$= B_\infty(V) + \frac{1}{c^2}E(V) \in \mathrm{SO}_c(1,n) \tag{16}$$

$V \in R_c^{n\times 1}$. Here the entanglement component (which entangles the space and the time of a single boosted particle) is $c^{-2}E(V)$, where

$$E(V) := \gamma_V \begin{pmatrix} \frac{\gamma_V}{1+\gamma_V}V^2 & V^t \\ \frac{\gamma_V}{1+\gamma_V}V^2 V & \frac{\gamma_V}{1+\gamma_V}VV^t \end{pmatrix}. \tag{17}$$

The additive decomposition (16) of the (1,n)-Lorentz boost $B_c(V)$ demonstrates that the effects of a (1,n)-Lorentz boost $B_c(V)$ are the sum of Galilean effects, due to the Galilean component $B_\infty(V)$, and of relativistic effects, due to the entanglement component $c^{-2}E(V)$. Furthermore, it demonstrates that the relativistic effects of a (1,n) -Lorentz boost are directly noticeable only at high speeds owing to the presence of the coefficient c^{-2} of $E(V)$ in (16).

The Galilean component of the additive decomposition (16) $B_c(V)$ is intuitively clear. Contrastingly, the entanglement part of the additive decomposition (16) $B_c(V)$ is counterintuitive, giving rise to relativistic effects like (i) entanglement of time and space of a boosted particle; (ii) time dilation; (iii) length contraction; (iv) Thomas precession; and (v) particle's energy levels.

Being intuitively clear, the Galilean component $B_\infty(V)$ of the additive decomposition (16) $B_c(V)$imparts the interpretation of V. It reveals the physical interpretation of the parameter $V \in \mathbb{R}_c^{n\times 1}$ that parametrizes the boost $B_c(V)\in\mathrm{SO}_c(1,n)$ as the velocity of a boosted particle relative to an inertial observer. In this sense, we say that the additive decomposition (16) enables *understanding Lorentz utilizing Galilei* in signature (m,n) for m = 1 and all $n\in N$.

We now face the task of *understanding Lorentz utilizing Galilei* in signature (m,n) for all m,n⊡N. To accomplish the task, we introduce the intuitively clear (m,n)-Galilei boost for all m,n⊡N. in Section 5. Completing this task will enable us to achieve the main goal of this paper, which is to demonstrate that the group $SO_c(m,n)$ of all (m,n)-Lorentz transformations is the symmetry group of multi-particle systems that consist of m n-dimensional entangled particles for all $m,n\in N$ (n = 3 in physical applications).

The (m,n)-lorentz boost additive decomposition

The (1,n)+-Lorentz boost additive decomposition is obtained in (16). Analogously, we now obtain the (m,n)-Lorentz boost additive decomposition for all $m,n\in N$.

Owing to Identities (5), the (m,n)-Lorentz boost in (7) possesses the remarkable *additive decomposition* as the sum of a Galilean component $B_\infty(V)$ and an entanglement component $c^{-2}E(V)$ given by

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix}$$

$$= \begin{pmatrix} I_m & 0_{m\times n} \\ V & I_n \end{pmatrix} + \frac{1}{c^2}\begin{pmatrix} \frac{(\Gamma_V^R)^2}{I_m+\Gamma_V^R}V^t V & \Gamma_V^R V^t \\ \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L}VV^t V & \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L}VV^t \end{pmatrix}$$

$$=: B_\infty(V) + \frac{1}{c^2}E(V) \in \mathrm{SO}_c(m,n) \tag{18}$$

for any $V \in \mathbb{R}_c^{n\times m}$.

The importance of the additive decomposition (18) rests on the fact that it enables the intuitively clear Galilean component $B_\infty(V)$ to impart interpretation to the counterintuitive (m,n) -Lorentz boost $B_c(V)$.

The additive decomposition (18) of signature (m,n) for all $m,n\in N$ extends the additive decomposition (16) of signature (m,n) for m = 1 all $n\in N$. It expresses the (m,n)-Lorentz boost $B_c(V)$ as the sum of the (m,n)-Galilei boost $B_\infty(V)$ and an entanglement component $c^{-2}E(V)$, where

$$B_\infty(V) = \begin{pmatrix} I_m & 0_{m\times n} \\ V & I_n \end{pmatrix}, \tag{19}$$

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118

$V \in \mathbb{R}^{n\times m}$, is the (m,n) -Galilei boost of signature (m,n) parametrized by V. We will find in Section 5 that the (m,n) -Galilei boost is intuitively clear and that its matrix parameter V is a velocity matrix for all $m,n\in N$. The m columns of V will turn out to be, respectively, the m velocities of m particles collectively boosted relative to an m inertial observer.

The effects of the entanglement component $c^{-2}E(V)$ are directly noticeable only at high speeds owing to the presence of the coefficient c^{-2} of $E(V)$ in (18). These effects entangle the m time and n space coordinates of m entangled n-dimensional particles. As such, the effects of the entanglement component $c^{-2}E(V)$ are counterintuitive relativistic effects that have to be confronted.

We now face the task of demonstrating that the Galilean component of the additive decomposition (18) is intuitively clear. This, in turn, will enable us to determine the physical interpretation of the matrix parameter $V \in \mathbb{R}_c^{n\times m}$ of the (m,n)-Lorentz boost, as well as the physical interpretation of the (m,n)-Lorentz boost itself.

Galilei boost of signature (m,n)

In the Galilean limit, $c\infty$, the (m,n) -Lorentz boost $B_c(V)$ in (7) tends to its associated (m,n)-Galilei boost $B_\infty(V)$,

$$\lim_{c\to\infty} B_c(V) = \begin{pmatrix} I_m & 0_{m\times n} \\ V & I_n \end{pmatrix} =: B_\infty(V) \in \mathrm{SO}_\infty(m,n), \tag{20}$$

$V\in\mathbb{R}^{n\times m}$, noting that $\lim_{c\to\infty}\Gamma_V^R = I_m$ and $\lim_{c\to\infty}\Gamma_V^L = I_n$

Here $\mathrm{SO}_\infty(m,n)$is the group of all -Galilei boosts for any $m,n\in N$.

To intuitively understand the (m,n)-Galilei boost $B_\infty(V)$ in (20), we consider its application to time-space coordinates in m-time and n-space dimensions.

The application of the (1,3) -Galilei boost to the time-space coordinates of a single particle in a position $\mathbf{x} = (x_1,x_2,x_3)^t$ at a time t is described in (15).

We now consider the Galilei boost of signature (m,n) for all $m,n\in N$, paying special attention to the case when $(m,n) = (2,3)$ as an illustrative example.

Let $B_\infty(V) = B_\infty(\mathbf{v}_1,\mathbf{v}_2)$ be the Galilei boost of signature (2,3), parametrized by the velocity matrix $V = (\mathbf{v}_1\ \mathbf{v}_2)$,

$$V = (\mathbf{v}_1\ \mathbf{v}_2) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{pmatrix} \in \mathbb{R}^{3\times 2}, \tag{21}$$

of two velocity vectors $\mathbf{v}_k = (v_{1k}, v_{2k}, v_{3k})^t\in\mathbb{R}^3$, $k = 1,2$. These two velocity vectors form the two columns of the velocity matrix V, in analogy with (14), where the velocity matrix V has a single column $\mathbf{v}$.

Furthermore, let

$$\begin{pmatrix} T \\ X \end{pmatrix} := \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \in \mathbb{R}^{5\times 2} \tag{22}$$

be a 5 × 2 matrix that represents a (2,3) -particle system. It is a multi-particle system consisting of two 3-dimensional particles, $(t_k,\mathbf{x}_k)$, $k = 1,2$, with positions $\mathbf{x}_k = (x_{1k},x_{2k},x_{3k})^t\in\mathbb{R}^3$, at time $t_k\in\mathbb{R}$, respectively. In general. An (m,n) -particle system is a multi-particle system consisting of m n -dimensional particles.

Here

$$T := \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \tag{23}$$

$t_1,t_2 > 0$, is a 2×2 diagonal matrix that represents the times t_1 and t_2 when two particles are observed at positions $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$, respectively; and

$$X := \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = (\mathbf{x}_1\ \mathbf{x}_2) \in \mathbb{R}^{3\times 2} \tag{24}$$

is a 3×2 matrix the columns of which represent the positions $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^3$ of two particles at times $t_1,t_2\in\mathbb{R}$, respectively.

Accordingly, the point $\begin{pmatrix} T \\ X \end{pmatrix} \in \mathbb{R}^{5\times 2}$ represents a (2,3)-particle system consisting of two particles $(t_1,\mathbf{x}_1)$ and $(t_1,\mathbf{x}_2)$ with positions $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$ at times t_1 and t_2, respectively.

The collective application of the Galilei boost $B_\infty(V)$ of signature (2,3) to the pair of particles $\begin{pmatrix} T \\ X \end{pmatrix}$ in $m+n = 2+3$ time-space dimensions yields

$$\begin{pmatrix} T' \\ X' \end{pmatrix} := B_\infty(V) \begin{pmatrix} T \\ X \end{pmatrix}, \tag{25}$$

which is described in the following chain of equations,

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118

$$\begin{pmatrix} t_1' & 0 \\ 0 & t_2' \\ \mathbf{x}_1' & \mathbf{x}_2' \end{pmatrix} := B_\infty(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ v_{11} & v_{12} & 1 & 0 & 0 \\ v_{21} & v_{22} & 0 & 1 & 0 \\ v_{31} & v_{32} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}$$
$$= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ v_{11}t_1 + x_{11} & v_{12}t_2 + x_{12} \\ v_{21}t_1 + x_{21} & v_{22}t_2 + x_{22} \\ v_{31}t_1 + x_{31} & v_{32}t_2 + x_{32} \end{pmatrix} \qquad (26)$$
$$= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 + \mathbf{v}_1 t_1 & \mathbf{x}_2 + \mathbf{v}_2 t_2 \end{pmatrix}.$$

Here, $B_\infty(V)$ in (25) – (26) is given by (19) with $m = 2$ and $n = 3$.

The chain of equations (26) describes the application of a Galilei boost $B_\infty(V)$ of signature (2,3) to collectively boost two particles, $(t_1,\mathbf{x}_1)$ and $(t_2,\mathbf{x}_2)$, into the two boosted particles, $(t_1,\mathbf{x}_1 + \mathbf{v}_1 t_1)$ and $(t_2,\mathbf{x}_2 + \mathbf{v}_2 t_2)$, by two 3-dimensional velocity vectors $\mathbf{v}_1 = (v_{11}, v_{21}, v_{31})^t$ and $\mathbf{v}_2 = (v_{12}, v_{22}, v_{32})^t$ in $\mathbb{R}^3$. It is important to note that the two collectively boosted particles are not entangled in the sense that the boost of each boosted particle is independent of the boost of the other boosted particle. Interestingly, this observation fails when we replace Galilei boosts of signature $(m,3)$, $m \geq 2$, with corresponding Lorentz boosts of the same signature $(m,3)$, as we will see in Section 6.

Each of the two particles $(t_1,\mathbf{x}_1)$ and $(t_1,\mathbf{x}_1)$ possesses a one-dimensional time, $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$, respectively. Accordingly, the system consisting of the two particles possesses the two-dimensional time, $(t_1,t_2) \in \mathbb{R}^2$. Each of the two particles possesses its clock so that the two-dimensional time of the system is measured by two clocks. In general, a multi-particle system consisting of m particles possesses an m -dimensional time, measured by m clocks, $m \in N$.

The extension of (21) – (26) from signature (2,3) to signature (m,n), for all $m,n \in N$, is now obvious. The Galilei boost $B_\infty(V)$ of signature (m,n) is parametrized by a velocity matrix $V \in \mathbb{R}^{n\times m}$ of order $n\times m$ that consists of m columns, $V = (\mathbf{v}_1\ \mathbf{v}_2 \ldots\ \mathbf{v}_m)$ that respectively represent the m velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ of m collectively boosted particles relative to an inertial observer. Furthermore, when $B_\infty(V)$ applied to collectively boost m particles in $\mathbb{R}^n$ (i) it keeps invariant each of the times t_k, $k = 1,\ldots,m$ of the m particles $(t_k,\mathbf{x}_k)$, that is, $t_k' = t_k$, and (ii) it boosts their positions $\mathbf{x}_k \in \mathbb{R}^n$ into the boosted positions $\mathbf{x}_k' = \mathbf{x}_k + \mathbf{v}_k t_k \in \mathbb{R}^n$ at times t_k, respectively. The m collectively boosted particles are not entangled in the sense that (i) the boost of each boosted particle is independent of the boosts and times of the other boosted particles and (ii) the time of each boosted particle is independent of the times and boosts of the other boosted particles.

A Galilei boost of signature $(m,3)$, applied *collectively* to the $m > 1$ particles of an $(m,3)$-particle system is thus equivalent to m Galilei boosts of signature $(1,3)$, applied *individually* to each particle of the system. Hence, a Galilei boost of signature (m,n), $m,n \geq 2$, can be viewed as a *Galilei multi-boost* acting on multi-particle systems. While Galilei multi-boosts involve no entanglement, we will see that corresponding Lorentz multi-boosts accommodate the entanglement of the space and time coordinates of multi-particle systems.

The chain of equations (26) for the action of Galilei boosts of signature (2,3) and its obvious extension to the action of Galilei boosts of any signature (m,n), $m,n \geq N$, demonstrates that the extension of the common Galilei boost of signature (1,3) to Galilei boosts of any signature (m,n) is quite natural and intuitively clear. The additive decomposition (18) provides a correspondence between Galilei boosts, $B_\infty(V)$, $V \in \mathbb{R}^{n\times m}$, of signature (m,n) and Lorentz boosts, $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, of the same signature (m,n). This correspondence indicates that the extension of the common Lorentz boost of signature (1,3) to Lorentz boosts of any signature (m,n) is quite natural as well, representing Lorentz multi-boosts when $m > 1$. Yet, unlike Lorentz boosts, Galilei boosts of signature (m,n) are intuitively clear.

Understanding Lorentz multi-boosts utilizing Galilei multi-boosts: The emergence of relativistic multi-particle entanglement

Following its introduction in Section 5, the (m,n)-Galilei multi-boost $B_\infty(V)$ is intuitively clear for any $(m,n) \in N$. As such, utilizing the additive decomposition (18) imparts interpretation to the counterintuitive (m,n)-Lorentz boost. Specifically, $B_\infty(V)$ revealed in (18) that the parameter $V \in \mathbb{R}_c^{n\times m}$ of the (m,n)-Lorentz boost $B_c(V)$ in (7) is a velocity matrix the m columns of which are respectively the m velocities of collectively m boosted particles relative to an inertial observer.

Accordingly, the interpretation that the (m,n)-Galilei boost imparts to the (m,n)-Lorentz boost utilizing the additive decomposition (18) is as follows: The (m,n)-Lorentz boost $m>1$ is a multi-boost that collectively boosts a multi-particle system of m n-dimensional particles by respective m velocities, which are the m columns of the velocity matrix $V \in \mathbb{R}_c^{n\times m}$. In this sense, we say that the additive decomposition (18) enables *understanding Lorentz utilizing Galilei*.

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118

Contrasting the intuitively clear Galilean component $B_\infty(V)$ of the additive decomposition (18), the entanglement component of the additive decomposition gives rise to counterintuitive relativistic effects for any $m>1$ and $n\in N$. These relativistic effects include (i) *relativistic entanglement* of the m-temporal and n-spatial coordinates of m collectively boosted particles; (ii) multi-time dilation; (iii) multi-length contraction; (iv) multi-time and space precession; and (v) multi-particle's energy levels.

In classical mechanics, the group $SO_\infty(m,3)$ of all $(m,3)$ -Galilei transformations (including Galilei boosts and rotations) is the symmetry group of any multi-particle system consisting of m particles. If we understand Lorentz by Galilei utilizing the additive decomposition (18), then Galilei imparts to Lorentz the following interpretation: In special relativistic mechanics the group $SO_c(m,3)$ of all $(m,3)$ -Lorentz transformations is the symmetry group of any multi-particle system consisting of m particles.

It is now clear why quantum entanglement involves Lorentz symmetry violation in special relativity theory, and how to confront the resulting problem. The symmetry group of m entangled particles, $m>1$, is not the standard Lorentz group $SO_c(m,3)$ of special relativity. Rather, the symmetry group of m entangled particles, $m>1$, is $SO_c(m,3)$.

A suggested search of experimental support for enriched special relativity theory

Stems from the Lorentz group $SO_c(m,3)$, special relativity theory does not admit particle entanglement. To enable special relativity to admit the entanglement of $m>1$ particles it is necessary to enrich it by incorporating the Lorentz groups $SO_c(m,3)$ for all $m>1$.

The resulting *enriched special relativity theory* thus stems from the Lorentz groups $SO_c(m,3)$ for all $m\in N$. It is hoped that this article will stimulate a search for experimental support for the necessity to enrich special relativity theory by incorporating the Lorentz groups $SO_c(m,3)$ for all $m>1$.

A suggested search for experimental support of our enriched special relativity theory follows: The shifting of energy levels that results from quantum entanglement, leading to Lorentz symmetry violation, is studied in [10]. While the shifting of energy levels violates $(1,3)$-Lorentz invariance, it perhaps obeys $(m_0,3)$-Lorentz invariance for some $m_0>1$. If m_0 exists, then the $(m_0,3)$-Lorentz invariance of the shifting of energy levels would provide experimental support for our enriched special relativity theory. Accordingly, a search for m_0 amounts to a search for a desired experimental support.

Conclusion

The (m,n)-Lorentz boost $B_c(V)$ in (7) is a Lorentz transformation of signature (m,n), $m,n\in N$, without rotations. It is a coordinate transformation of m temporal coordinates and n spatial coordinates of $(m+n)$-dimensional spacetime, which leaves invariant the squared pseudonorm (1). In the special case when $m = 1$ $B_c(V)$ descends to the common $(1,n)$-Lorentz boost of special relativity theory (SRT), where $n = 3$ in physical applications.

In the Newtonian limit, $c\to\infty$, the counterintuitive (m,n)-Lorentz boost $B_c(V)$ tends to the intuitively clear (m,n)-Galilei boost $B_\infty(V)$ in (20). The (m,n)-Galilei boost turns out to be a multi-boost, that is, a boost that boosts simultaneously m n-dimensional particles simultaneously.

The (m,n)-Lorentz boosts and the (m,n) -Galilei boosts are related by the additive decomposition (16) for $m = 1$ and (18) for $m\geq 1$. Employing the additive decomposition, the intuitively clear (m,n)-Galilei boost imparts interpretation to the (m,n)-Lorentz boost, revealing that the latter, like the former, is a multi-boost, simultaneously boosting m n-dimensional particles.

Finally, it is clear from the additive decomposition (18) that Galilei multi-boosts admit no entanglement while, in contrast, Lorentz multi-boosts admit entanglement. Hence, to enable SRT to admit entanglement of m 3-dimensional particles it seems to be useful to enrich SRT by incorporating into SRT the $(m,3)$-Lorentz groups for all $m>1$. A search for experimental support for enriched SRT in terms of the shifting of energy levels that result from quantum entanglement is proposed in Section 8.

References

1. Ahn D, Lee HJ, Hwang SW, Kim MS. Is quantum entanglement invariant in special relativity? [Internet]. 2003 [cited 2024 Jun 3].
2. Alsing PM, Fuentes I. Observer-dependent entanglement. Classical and Quantum Gravity. 2012 Nov 23;29(22):224001.
3. Caban P, Hiesmayr BC. Bound entanglement is not Lorentz invariant. Sci Rep. 2023 Jul;13.
4. Gingrich RM, Adami C. Quantum entanglement of moving bodies. Phys Rev Lett. 2002 Dec 30;89(27):270402. doi: 10.1103/PhysRevLett.89.270402. Epub 2002 Dec 16. PMID: 12513186.
5. Li L, Li X, Zhang B, You L. Enhancing test precision for local Lorentz-symmetry violation with entanglement. Phys Rev A. 2019 Apr;99(4):042118.
6. Rembieliński J, Smoliński KA. Quantum preferred frame: Does it really exist? EPL (Europhysics Letters). 2009 Oct 2; 88(1):10005.
7. KLOE-2 Collaboration. Test of CPT and Lorentz symmetry in entangled neutral kaons with the KLOE experiment. Phys Lett B. 2014; 730:89-94.
8. Di Domenico A. Search for CPT and Lorentz-symmetry violation in entangled neutral kaons. In: Proceedings of the Seventh Meeting on CPT and Lorentz Symmetry; 2017. World Scientific. 97-100.
9. Dzuba VA, Flambaum VV, Safronova MS, Porsev SG, Pruttivarasin T, Hohensee MA, Haffner H. Strongly enhanced effects of Lorentz symmetry violation in entangled Yb+ ions. Nat Phys. 2016 May;12(5):465-468.

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118

10. Shaniv R, Ozeri R, Safronova MS, Porsev SG, Dzuba VA, Flambaum VV, Haffner H. New methods for testing Lorentz invariance with atomic systems. Phys Rev Lett. 2018 Mar 9;120(10):103202.

11. Shi Y. Some exact results on CP and CPT violations in a C=-1 entangled pseudoscalar neutral meson pair. Eur Phys J C. 2013; 73:2506.

12. Silarski M, KLOE-2 Collaboration. CPT and Lorentz symmetry tests with entangled neutral kaons at KLOE/KLOE-2. J Phys Conf Ser. 2014; 556:012071.

13. Zhuang M, Huang J, Lee C. Entanglement-enhanced test proposal for local Lorentz-symmetry violation via spinor atoms. arXiv preprint arXiv:2201.11366. 2022.

14. Horodecki R, Horodecki P, Horodecki M, Horodecki K. Quantum entanglement. Rev Mod Phys. 2009 Apr;81(2):865-942.

15. Hamermesh M. Group theory and its application to physical problems. Addison-Wesley Ser Phys. Reading, Mass.-London: Addison-Wesley Publishing Co., Inc.; 1962.

16. Ungar AA. Beyond Pseudo-Rotations in Pseudo-Euclidean Spaces: An introduction to the theory of bi-gyrogroups and bi-gyrovector spaces. Math Anal Appl. Elsevier/Academic Press; 2018.

17. Horn RA, Johnson CR. Matrix analysis. Cambridge: Cambridge University Press; 1990.

Citation: Ungar AA (2024) Understanding Lorentz Utilizing Galilei: The Emergence of a Friendly Extended Special Relativity Theory that Admits Relativistic Multi-Particle Entanglement. Ann Math Phys 7(2): 150-157. DOI: https://dx.doi.org/10.17352/amp.000118