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Research Article

# The Riemann's Hypothesis, the Prime Numbers Theorem (PNT), and the Error

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## Abstract

In this simple paper, a small refinement to the Prime Number Theorem (PNT) is proposed, which allows us to limit the error with which said theorem predicts the value of the Prime-counting function π(x); and, in this way, endorse the veracity of the Riemann Hypothesis.

Many people know that the Riemann Hypothesis is a difficult mathematical problem - even to understand - without a certain background in mathematics. Many techniques have been used, for more than 150 years, to try to solve it. Among them is the one that establishes that, if the Riemann hypothesis is true, then the error term that appears in the prime number theorem can be bounded in the best possible way. Specifically, Helge von Koch demonstrated in 1901 that it should be:

$$\Pi\left(x\right) = Li\left(x\right) + O(\sqrt{x}\ \ln x)$$

A refined variant of Koch's result, given by Lowell Schoenfeld in 1976, states that the Riemann hypothesis is equivalent to the following result:z

$$\frac{dX}{dt} = -k\left(1-X\right)^{n} \quad \text{for all } x \geq 2657$$

In this work, we will make a small adjustment to the prime counting function π(x), which will allow us to limit the error with which the PNT predicts the value of π(x) to a value even smaller than that established by Lowell Schoenfeld.

## 1.1 Introduction

The history of mathematics is full of examples where conjectures were established based on intuition, or simply based on numerical research. An example of this, very relevant in this case, is the famous prime number theorem (PNT), initially formulated by Gauss in 1792 [1-6]. In this work, we will make a small adjustment to the prime counting function π(x), which not only improves the estimate of the number of primes up to x but will also allow us to limit the error with which the PNT predicts the value of π(x).

## 1.2 Previous concepts

The prime number theorem states that $\pi\left(x\right) \approx \frac{x}{\ln\left(x\right)}$ (1)

Where π (x) is the number of primes smaller than x , and ln (x) is the natural logarithm of x.

## 1.3 New estimate for π(x)

At (1), we use $x = n^2 + 2n + 1 = (n+1)^2$; then is:



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$$\pi\left((n+1)^2\right) \approx \frac{n^2+2n+1}{\ln\left(n^2+2n+1\right)} l = \frac{n^2}{\ln\left((n+1)^2\right)} + \frac{2n+1}{\ln\left((n+1)^2\right)};$$

where $\frac{n^2}{\ln\left((n+1)^2\right)}$ is:

$$\frac{n^2}{\ln\left((n+1)^2\right)} = \frac{n^2}{2\ln\left((n+1)\right)} = \frac{n}{2}\cdot\frac{n}{\ln\left(n+1\right)} \approx \frac{n}{2}\cdot\frac{n}{\ln\left(n\right)} \approx \pi\left(n^2\right)$$

In 1896, de la Vallée Poussin [7] showed that x/(ln x-a) provides a better approximation to π(x) than x/(ln x), and also showed that using a=1 is the best option. So, we replace, in the previous equation, $\frac{n}{\ln\left(n+1\right)}$ by $\frac{n}{\ln\left(n+1-1\right)} \approx \pi\left(n\right)$, being $\frac{n}{2}.\pi\left(n\right) \approx \pi\left(n^2\right)$

therefore, it is $\pi(n+1)^2 \approx \pi\left(n^2\right) + \frac{2n+1}{\ln\left((n+1)^2\right)}$ (2)

Before using expression (2), to address the issue of the error with which the PNT predicts the value of π(x), we are going to show how said expression (2) can be used to improve the estimate of the number of primes up to x, given by the PNT.

We start from n=2, with $\Pi(2^2)=2$ (primes 2 and 3)

$$\pi\left(3^2\right) \approx \pi\left(2^2\right) + \frac{2x2+1}{\ln\left(3^2\right)} = 2 + \frac{5}{2\ln 3}$$

$$\pi\left(4^2\right) \approx \pi\left(3^2\right) + \frac{2x3+1}{\ln\left(4^2\right)} = 2 + \frac{5}{2\ln 3} + \frac{7}{2\ln\left(4\right)}$$

$$\pi\left(5^2\right) \approx \pi\left(4^2\right) + \frac{2x4+1}{\ln\left(5^2\right)} = 2 + \frac{5}{2\ln 3} + \frac{7}{2\ln\left(4\right)} + \frac{9}{2\ln\left(5\right)}$$

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$$\pi\left((n+1)^2\right) \approx \pi\left(n^2\right) + \frac{2n+1}{\ln\left(n+1\right)^2} = 2 + \frac{5}{2\ln 3} + \frac{7}{2\ln\left(4\right)} + \frac{9}{2\ln\left(5\right)} + ..... + \frac{2n+1}{2\ln\left(n+1\right)}$$

$$\pi\left(n^2\right) \approx 2 + \frac{1}{2}\sum_{i=3}^{n}\frac{2i-1}{\ln(i)} \qquad (3)$$

If in equation (2) we replace $n^2$ by $x$ we get:

$$\pi\left(x+2\sqrt{x}+1\right) \approx \pi\left(x\right) + \frac{2\sqrt{x}+1}{\ln\left(x+2\sqrt{x}+1\right)} \qquad (2.1)$$

Or generalizing, we could write:

$$\pi\left(x+\Delta x\right) \approx \pi\left(x\right) + \frac{\Delta x}{\ln\left(x+\Delta x\right)} \qquad (2.2)$$

If $x$ is not a square number, then the expression (3) will be:

$$\pi\left(x\right) \approx 2 + \frac{1}{2}\sum_{i=3}^{\sqrt{x}}\frac{2i-1}{\ln(i)} + \frac{x-\lfloor\sqrt{x}\rfloor^2}{\ln(x)} \qquad (3.1)$$

The actual values of π(x), the values of π(x) estimated with the PNT, and with equation (3), and the relative error of (3), are shown, for various values of x, in the following table 1.

It should be noted that in this work we have not modified the PNT but rather the way in which we use it. For example, to apply it to a value X, we do not use X/ln(X) directly, but we arrive at X by applying said theorem several times ($\sqrt{x}$ -1 times) as indicated by equation (3); or in a single step - knowing the value of $\pi\left(\left(n^2\right)\right)$ - to find $\left(\pi\left((n+1)^2\right)\right)$ as indicated by equation (2).

Now we are going to address the issue of the error with which the PNT approximates the value of π(x); we focus on expression (2).

$$\pi(n+1)^2 \approx \pi\left(n^2\right) + \frac{2n+1}{\ln\left((n+1)^2\right)} \qquad (2)$$

- We will assume that between $n^2$ y $(n+1)^2$ equation (2) will approximate the number of primes with the greatest possible error. This can happeCase 1: when between $n^2$ and $(n+1)^2$ the density of primes is the minimum possible.
- Case 2: when between $n^2$ and $(n+1)^2$ the density of primes is the maximum possible.

**Case 1:**

Legendre's conjecture [8], proposed by Adrien-Marie Legendre states that there is always at least one prime number between $n^2$ and $(n+1)^2$.

Equation (2) estimates that between $n^2$ and $(n+1)^2$ there are $\frac{2n+1}{\ln\left((n+1)^2\right)}$ primes, therefore, when the density of primes is the minimum possible the error will be:

$$E = \frac{2n+1}{\ln\left((n+1)^2\right)} - 1$$

**Case 2:**

Equation (2) estimates that

$$\pi(n^2+2n+1) - \pi\left(n^2\right) \approx \frac{2n+1}{\ln\left((n+1)^2\right)} = \frac{1}{2}\frac{2n+1}{\ln\left(n+1\right)}$$



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**Table 1:** compare the actual value of π(x), with π(x) estimated by PNT and by equation (3).

| $x$ | $\Pi(x)$ | $\frac{x}{\ln(x)-1}$ | $2+\frac{1}{2}\sum_{i=3}^{n}\frac{2i-1}{ln(i)}$ | % error between $\Pi(x)$ and $2+\frac{1}{2}\sum_{i=3}^{n}\frac{2i-1}{ln(i)}$ |
|---|---|---|---|---|
| 2^6 | 18 | 20 | 19 | 5,55% |
| 2^8 | 54 | 56 | 57 | 5,55% |
| 2^10 | 172 | 172 | 177 | 2,90% |
| 2^12 | 564 | 559 | 572 | 1,41% |
| 2^14 | 1900 | 1882 | 1913 | 0,68% |
| 2^16 | 6542 | 6495 | 6575 | 0,50% |
| 2^18 | 23000 | 22.841 | 23056 | 0,24% |
| 2^20 | 82025 | 81519 | 82119 | 0,11% |
| 2^22 | 295947 | 294353 | 296086 | 0,047% |
| 2^24 | 1077871 | 1073018 | 1078178 | 0,028% |
| 2^26 | 3957809 | 3942518 | 3958278 | 0,0118% |
| 2^28 | 14630843 | 14582447 | 14631660 | 0,00558% |
| 2^30 | 54400028 | 54244684 | 54401277 | 0,00229% |
| 2^32 | 203280221 | 202777307 | 203283743 | 0,00173% |
| 2^34 | 762939111 | 761282670 | 762943858 | 6,22 e-4% |
| 2^36 | 2874398515 | 2868894100 | 2874411028 | 4,35 e-4% |
| 2^38 | 10866266172 | 10847763358 | 10866287179 | 1,93 e-4% |
| 2^40 | 41203088796 | 41140322812 | 41203127190 | 9,318 e-5% |
| 2^42 | 156661034233 | 156446289948 | 156661087433 | 3,395 e-5% |
| 2^44 | 597116381732 | 596376100156 | 597116504059 | 2,0486 e-5% |
| 2^46 | 2280998753949 | 2278428606753 | 2280998901761 | 6,480 e-6% |
| 2^48 | 8731188863470 | 8722209186967 | 8731189492312 | 7,20 e-6% |
| 2^50 | 33483379603407 | 33451819731490 | 33483380608016 | 3,00 e-6% |
| 2^52 | 128625503610475 | 128513987322140 | 128625504950092 | 1,041 e-6% |
| 2^54 | 494890204904784 | 494494217586814 | 494890210114268 | 1,05 e-6% |
| 2^56 | 1906879381028850 | 1905466805129420 | 1906879387367592 | 3,324 e-7% |
| 2^58 | 7357400267843990 | 7352339978156042 | 7357400282019924 | 1,926 e-7% |
| 2^60 | 28423094496953330 | 28404895655494853 | 28423094518060616 | 7,426 e-8% |

But, now assuming that between $n^2$ and $(n+1)^2$ the density of primes will be the maximum possible, we double the previous result supposing that:

$$\pi(n^2+2n+1)-\pi\left(n^2\right)\approx 2\left(\frac{1}{2}\frac{2n+1}{\ln\left(n+1\right)}\right)=\frac{2n+1}{\ln\left(n+1\right)}$$

However, $\frac{2n+1}{\ln\left(n+1\right)}>\frac{2n+1}{\ln\left(2n+1\right)}$; the latter being the estimate for $\pi(2n+1)$ made by the PNT, taking those $(2n+1)$ places from the origin of coordinates.

Therefore, the maximum density of primes assumed between $n^2$ and $(n+1)^2$, exceeds reality, since unlike what happens from the origin, after $n^2$ all primes ≤ $n$ appear spreading their multiples in the gap given by 2n+1. So, the error, in case 2, will be less than the additional of

$$\frac{1}{2}\frac{2n+1}{\ln\left(n+1\right)}=\frac{2n+1}{\ln\left((n+1)^2\right)}.$$

Now, unifying the treatment of cases 1 and 2, we say that the error in the counting of primes - approximated by the PNT (using equation (2))- is, in absolute value, the following:

$$\text{E}\leq\frac{2n+1}{\ln\left((n+1)^2\right)}-1 \qquad (4)$$

On the other hand, we know that Helge von Koch



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demonstrated in 1901 that, if and only if the Riemann hypothesis holds, it is:

$$\Pi(x) = Li(x) + O(\sqrt{x}\ln x)$$

A refined variant of Koch's result, given by Lowell Schoenfeld in 1976, states that the Riemann hypothesis is equivalent to the following result:

$$E < \frac{1}{8\pi}\sqrt{x}\ln(x),\text{ , for each } x \geq 2657 \quad (5)$$

In equation (4) we make

$$E \leq \frac{2n+1}{\ln\left((n+1)^2\right)} - 1 = \frac{2n}{\ln\left((n+1)^2\right)} + \frac{1}{\ln\left((n+1)^2\right)} - 1 \;\therefore\text{ is:}$$

$$E < \frac{2n}{\ln\left((n+1)^2\right)}\text{, here we make } x = (n+1)^2 \quad \therefore \quad \text{is:}$$

$$n+1 = \sqrt{x} \Rightarrow 2n = 2\sqrt{x} - 2$$

$$E < \frac{2\sqrt{x}-2}{\ln(x)} \Rightarrow E\frac{2\sqrt{x}}{\ln(x)} \quad (4.1)$$

Finally, we compare (4.1) and (5)

$$E < \frac{2\sqrt{x}}{\ln(x)} < \frac{1}{8\pi}\sqrt{x}\ln(x)$$

So, we have that:

$$\sqrt{x} < \frac{1}{16\pi}\ln(x)\ln(x)\sqrt{x} \Rightarrow \frac{1}{16\pi}\ln(x)\ln(x) > 1,$$

Therefore, it is: $\ln(x) > \sqrt{16\pi} = 4\sqrt{\pi} \Rightarrow x > 1200$

This means that the statement made by Lowell Schoenfeld in 1976, that the Riemann hypothesis is equivalent to proving that PNT approximates the value of $\pi(x)$ with an error $E < \frac{1}{8\pi}\sqrt{x}\ln(x),$ for all $x \geq 2657$, it is satisfied since $x \geq 1200$. Thus, this work reinforces and shows the veracity of the RH.

## Discussion and conclusion

According to the Riemann hypothesis, the density of primes decreases according to the prime number theorem (PNT). The PNT determines the average distribution of the primes, and the Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

This work aims to prove that the Riemann Hypothesis (RH) is true, and this would have far-reaching consequences for number theory and the use of primes in cryptography. For example (assuming RH), the Miller–Rabin primality test [9], is guaranteed to run in polynomial time.

## References


1. Petersen BE. Prime number theorem. 1996. Available from: https://www.math.ucdavis.edu/~tracy/courses/math205A/PNT_Petersen.pdf
2. Bombieri E. Problems of the millennium: the Riemann hypothesis. 2000. Available from: https://www.claymath.org/wp-content/uploads/2022/05/riemann.pdf
3. Titchmarsh EC, Heath-Brown DR. The theory of the Riemann zeta-function. 1986. Clarendon Press, University of Oxford. Available from: https://sites.math.rutgers.edu/~zeilberg/EM18/TitchmarshZeta.pdf
4. von Koch H. Contribution à la théorie des nombres premiers. Acta Math. 1910;33:293–320. Available from: https://doi.org/10.1007/BF02393216
5. Schoenfeld L. Available from: https://academia-lab.com/enciclopedia/lowell-schoenfeld/
6. Goldstein LJ. A history of the prime number theorem. Am Math Monthly. 1973;80(6):599–615. Available from: https://www.math.fsu.edu/~quine/ANT/2010%20Goldstein.pdf
7. De la Vallée Poussin. Cap III: The theorem of Hadamard and de la Vallée Poussin, and its consequences. Available from: https://sites.math.rutgers.edu/~zeilberg/EM18/TitchmarshZeta.pdf
8. Correa C. Gap between consecutive coprimes and gap between consecutive primes. Available from: https://www.academia.edu/23707567/Gap_between_consecutive_coprimes_and_Gapbetween_consecutive_primes
9. Narayan S, Corwin D. Improving the speed and accuracy of the Miller-Rabin primality test. MIT Primes USA. 2015. Available from: https://math.mit.edu/research/highschool/primes/materials/2014/Narayanan.pdf




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