Peertechz

Annals of Mathematics and Physics

submitpaper@peertechz.com

MATHEMATICS AND PHYSICS GROUP

ANNALS OF Mathematics and Physics OPEN ACCESS JOURNAL

ISSN: 2689-7636 DOI: https://dx.doi.org/10.17352/amp

Short Communication

Rayleigh Quotient and Surjectivity of Nonlinear Operators in Hilbert space

Raffaele Chiappinelli*

Department of Information Engineering and Mathematical Sciences, University of Siena, 53100 Siena, Italy

Received: 23 September, 2024
Accepted: 27 September, 2024
Published: 28 September, 2024

***Corresponding author:** Raffaele Chiappinelli, Department of Information Engineering and Mathematical Sciences, University of Siena, 53100 Siena, Italy, E-mail: raffaele.chiappinelli@unisi.it

Keywords:: Eigenvalue of a nonlinear operator; Gradient operator; Ekeland variational principle

Abstract

We consider continuous operators acting in a real Hilbert space and indicate conditions ensuring their continuous invertibility and/or surjectivity. In the case of bounded linear operators, these facts are well-known from basic Functional Analysis. The objective of this work is to indicate how similar properties can be proved also when the operators are not necessarily linear, using as a main tool their Rayleigh quotient and especially its lower and upper bound. In particular, we focus our attention on gradient operators and show a quantitative criterion that ensures their surjectivity through the positivity of an additional constant related to the measure of noncompactness.

This short work does not contain any detailed discussion or proof, but merely a few statements and comments concerning some properties of nonlinear (= not necessarily linear) operators acting in a Hilbert space; it aims at inviting people interested in the subject to further study the matter. References for proofs of the results are given throughout. Thus let H be a real Hilbert space with scalar product denoted $\langle .,. \rangle$ and corresponding norm $\| x \| = \sqrt{\langle x, x \rangle}$. If F is any map of H into itself, it makes sense to define its *Rayleigh quotient* by the formula

$$\frac{\langle F(x), x\rangle}{\| x \|^2} \ (x \in H, x \neq 0). \tag{1}$$

If we suppose in addition that F is continuous and such that

$$\| F(x) \| \leq A \| x \| \tag{2}$$

for some $A \geq 0$ and all $x \in H$, then its Rayleigh quotient is a bounded continuous real function, and we look in particular at the numbers

$$m(F) = \inf_{x \neq 0} \frac{\langle F(x), x\rangle}{\| x \|^2}, \ M(F) = \sup_{x \neq 0} \frac{\langle F(x), x\rangle}{\| x \|^2} \tag{3}$$

which are quite useful in the study of the spectral properties of F. Indeed it is immediate that if λ is an eigenvalue of F (meaning that $F(x) - \lambda x = 0$ for some $x \neq 0$), then $m(F) \leq \lambda \leq M(F)$. Moreover in the special case that $F=T$, a bounded linear operator, then the whole spectrum $\sigma(T)$ of T satisfies the inclusion

$$\sigma(T) \subset [m(T), M(T)] \tag{4}$$

as follows for instance using the Lax-Milgram Lemma (see, e.g., [1]). More can be said if T is in addition self-adjoint and/or compact; and quite surprisingly, similar interesting properties can be drawn also when T is replaced by a nonlinear operator F acting in H. For instance, if F is Lipschitz continuous and satisfies the condition

$$m_0(F) \equiv \inf_{x \neq y} \frac{\langle F(x) - F(y), x - y\rangle}{\| x - y \|^2} > 0, \tag{5}$$

then F is a *Lipeomorphism* (in the language of [2]), in the sense that it is a Lipschitz homeomorphism of H onto itself with Lipschitz inverse F^{-1}: as for surjectivity, this follows easily from the Minty-Browder Theorem (see, e.g., [1]).

Something can be said also in case F, rather than being Lipschitzian, satisfies the weaker condition (2): for if we put $b(F) = \inf_{x\neq 0} \frac{\|F(x)\|}{\|x\|}$ and

$$\Sigma_b(F) = \{\lambda \in \mathbb{R} : b(F - \lambda I) = 0\} \quad (6)$$

(a sort of ``approximate point spectrum" of F), then we have $\Sigma_b(F) \subset [m(F), M(F)]$. Finally, some surjectivity properties of F can be derived through the numbers $m(F)$, $M(F)$ at least in the case F is a *gradient* operator, i.e., is such that

$$\langle F(x), y\rangle = f'(x)y \quad (x, y \in H) \quad (7)$$

for some differentiable functional $f : H \to \mathbb{R}$; here $f'(x)$ denotes the (Fréchet) derivative of f at the point $x \in H$. Indeed using the Ekeland Variational Principle (see, e.g., [3]) we can show that for such an operator, the conditions

$$m(F)>0 \text{ and } \omega(F)>0 \quad (8)$$

where

$$\omega(F) = \inf\{\frac{\alpha(F(A))}{\alpha(A)} : A \subset E, A \text{bounded}, \alpha(A) > 0\} \quad (9)$$

and $\alpha(A)$ denotes the measure of non-compactness of the bounded set $A \subset H$, ensure that F is surjective. This implies in particular the surjectivity of $F-\lambda I$ when

$$\lambda \notin [m(F), M(F)] \cup \sigma_\omega(F), \quad (10)$$

where $\sigma_\omega(F) \equiv \{\lambda \in \mathbb{R} : \omega(F - \lambda I) = 0\}$.

Proofs of these statements can be found in [4-6], while we refer to [2] for a general introduction to the subject and also as a reference for further study.

Communication was held at the Conference on Topological Methods for Nonlinear Analysis and Dynamical Systems *(Firenze, 27-28 September 2024) organized in honor of the retirement of Professor Patrizia Pera.*

References

1. Brezis H. Functional Analysis, Sobolev Spaces and Partial Differential Equations. New York: Springer; 2011. Available from: https://link.springer.com/book/10.1007/978-0-387-70914-7
2. Appell J, De Pascale E, Vignoli A. Nonlinear Spectral Theory. Berlin: de Gruyter; 2004. Available from: https://doi.org/10.1515/9783110199260
3. de Figueiredo DG. Lectures on the Ekeland Variational Principle with Applications and Detours. Bombay: Tata Institute of Fundamental Research; 1989. Available from: https://mathweb.tifr.res.in/sites/default/files/publications/ln/tifr81.pdf
4. Chiappinelli R. Surjectivity of coercive gradient operators in Hilbert space and nonlinear spectral theory. Ann Funct Anal. 2019;10(2):170-179. Available from: http://dx.doi.org/10.1215/20088752-2018-0003
5. Chiappinelli R, Edmunds DE. Measure of noncompactness, surjectivity of gradient operators and an application to the p-Laplacian. J Math Anal Appl. 2019;471:712-727.
Available from: https://doi.org/10.1016/j.jmaa.2018.11.010
6. Chiappinelli R, Edmunds DE. Remarks on surjectivity of gradient operators. Mathematics. 2020;8:1538.
Available from: https://doi.org/10.3390/math8091538

Citation: Chiappinelli R. Rayleigh Quotient and Surjectivity of Nonlinear Operators in Hilbert space. Ann Math Phys. 2024;7(3):277-278.
Available from: https://dx.doi.org/10.17352/amp.000132