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Research Article

Simpson Type Estimations for Convex Functions via Quantum Calculus

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Abstract

We first establish a new identity including quantum integrals and quantum numbers via q -differentiable functions. After that, with the help of this equality, a Simpson-type inequality for functions whose quantum derivatives in modulus are convex is derived, and some new inequalities for powers of quantum derivatives in absolute value are provided. It is also discussed how results come out in the case when q approaches 1.

Mathematics Subject Classification: 26D15, 26D10, 26A51, 34A08.

Introduction

Simpson's rules (Thomas Simpson 1710-1761) are well-known methods in numerical analysis for the purpose of numerical integration and the numerical approximation of definite integrals. Two famous Simpson's rules are known in the literature, and one of them is the following estimation known as Simpson's inequality:

Theorem 1 *Suppose that* $\chi:[\rho,\sigma]\to\mathbb{R}$ *is a four times continuously differentiable mapping on* (ρ, σ), *and let* $\left\|\chi^{(4)}\right\|_{\infty}=\sup_{\varkappa\in(m,n)}\left|\chi^{(4)}(\varkappa)\right|<\infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\chi(\rho)+\chi(\sigma)}{2}+2\chi\left(\frac{\rho+\sigma}{2}\right)\right]-\frac{1}{\sigma-\rho}\int_{\rho}^{\sigma}\chi(\varkappa)d\varkappa\right|$$

$$\leq\frac{1}{2880}\left\|\chi^{(4)}\right\|_{\infty}(\sigma-\rho)^{4}.$$

Simpson's inequalities have been extensively studied by numerous researchers due to their wide range of applications in various fields of mathematics. Simpson's inequalities for different classes of functions have been studied, but one can find many inequality papers in the literature based on convex functions since convex functions are the basis of Simpson's inequality for integrals. For example, Alomari, et al. introduced some inequalities of Simpson's type based on functions whose absolute value of the first derivative is s-convex and concave in [1]. New inequalities of Simpson type for functions of bounded variation and their application to quadrature formulae in Numerical Analysis are given by Dragomir, et al. in [2]. In [3], Sarıkaya, et al. generalized the inequalities based on s-convex functions given by Alomari. In [4], Sarıkaay, et al. derived recent inequalities of Simpson type involving local fractional integrals for generalized convex functions are derived. What's more, An inequality of the Simpson type for an n-times continuously differentiable mapping is given by Liu in [5]. In addition to the references mentioned here, there are many articles on Simpson's inequality in the literature. Interested readers can find papers on Simpson-type inequalities in the literature for any class of function.

On the other hand, many studies have recently been carried out in the field of q -analysis, starting with Euler due to the high demand for mathematics that models quantum

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computing q -calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions, and other sciences quantum theory, mechanics, and the theory of relativity [6-12]. Apparently, Euler was the founder of this branch of mathematics, by using the parameter q in Newton's work of infinite series. Later, Jackson was the first to develop q -calculus that was known without limits calculus in a systematic way [9]. In 1908-1909, Jackson defined the general q -integral and q -difference operator [11]. In 1969, Agarwal described the q -fractional derivative for the first time [13]. In 1966-1967 Al-Salam introduced a q -analogue of the Riemann-Liouville fractional integral operator and q-fractional integral operator [14]. In 2004, Rajkovic gave a definition of the Riemann-type q -integral which was generalized of Jackson q-integral. In 2013, Tariboon introduced ${}_{\rho}D_q$ -difference operator [15].

Many integral inequalities well known in classical analysis such as Hölder inequality, Hermite-Hadamard inequality, Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev, and other integral inequalities have been proved and applied for q -calculus using classical convexity. For illustrate, Alp, et al. proved The fundamental q -Hermite–Hadamard inequality, some new q -Hermite–Hadamard inequalities, and generalized q -Hermite–Hadamard inequality for convex and quasi-convex functions in [16]. In addition, Noor, et al. investigated some new integral inequalities including q -integrals related to Hermite-Hadamard and Ostrowski-type integral inequalities by using different classes of mappings [17-19]. In [20], via newly defined quantum integrals, Simpson and Newton-type inequalities for convex functions are established by Budak, et al.

In light of all these studies, Simpson-type inequalities involving quantum integrals for functions whose absolute value of the first derivative is convex will be analyzed in this work. Actually, Tunç, et al. [21] obtained Simpson's type quantum integral inequalities. Unfortunately, there are many mistakes in the proofs. For example, in Lemma 4, Tunç found the equality

$$\int_0^{\frac12}(1-\tau)\left|q\tau-\frac16\right|{}_0d_q\tau$$

$$=\int_0^{\frac12}\left|q\tau-\frac16\right|{}_0d_q\tau-\int_0^{\frac12}\tau\left|q\tau-\frac16\right|{}_0d_q\tau$$

$$=\int_0^{\frac{1}{6q}}\left(q\tau-\frac16\right){}_0d_q\tau+\int_{\frac{1}{6q}}^{\frac12}\left(\frac16-q\tau\right){}_0d_q\tau$$

$$-\left(\int_0^{\frac{1}{6q}}\tau\left(q\tau-\frac16\right){}_0d_q\tau+\int_{\frac{1}{6q}}^{\frac12}\tau\left(\frac16-q\tau\right){}_0d_q\tau\right).$$

Here, for q (0, 1), $\frac{1}{6q}\not\leq\frac12$. For instance, $q=\frac16\to 1\not\leq\frac12$. So, the proof of Lemma 4 is not correct. Lemma 5 also has the same errors. On the other hand, since Lemma 4 and Lemma 5 are used in the proof of Theorem 1, there are errors in this theorem. Moreover, Theorem 2 and 3 have the same mistakes. For instance, because of (9), the following equalities are also not true:

$$\int_0^{\frac12}\left|q\tau-\frac16\right|^p{}_0d_q\tau=\frac{\left(1+(3q-1)^{p+1}\right)(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)},$$

$$\int_{\frac12}^{1}\left|q\tau-\frac56\right|^p{}_0d_q\tau=\frac{\left[(5-3q)^{p+1}+(6q-5)^{p+1}\right](1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}.$$

The integral boundaries that cause all these errors are chosen independently of q

In 2018 Tunç, et al. [21] obtained Simpson's type quantum integral inequalities. Unfortunately, there are many mistakes in the proofs. Many q -integrals are calculated incorrectly. Besides, the results of the lemma and theorems are also wrong. For example, in Lemma 4,

$$\int_0^{\frac12}(1-\tau)\left|q\tau-\frac16\right|{}_0d_q\tau=\int_0^{\frac{1}{6q}}\left(q\tau-\frac16\right){}_0d_q\tau+\int_{\frac{1}{6q}}^{\frac12}\left(\frac16-q\tau\right){}_0d_q\tau$$

$$-\left(\int_0^{\frac{1}{6q}}\tau\left(q\tau-\frac16\right){}_0d_q\tau+\int_{\frac{1}{6q}}^{\frac12}\tau\left(\frac16-q\tau\right){}_0d_q\tau\right).$$

Here, for $q\in(0,1)$, $\frac{1}{6q}\not\leq\frac12$. For instance, $q=\frac16\to 1\not\leq\frac12$. So,

the proof of Lemma 4 is not correct. Lemma 5 also has the same errors. On the other hand, since Lemma 4 and Lemma 5 are used in the proof of Theorem 1, there are errors in this theorem. Moreover, Theorem 2 and 3 have the same mistakes. For instance, because of (9), the following equalities are also not true:

$$\int_0^{\frac12}\left|q\tau-\frac16\right|^p{}_0d_q\tau=\frac{\left(1+(3q-1)^{p+1}\right)(1-q)}{6^{p+1}q\left(1-q^{p+1}\right)},$$

$$\int_{\frac12}^{1}\left|q\tau-\frac56\right|^p{}_0d_q\tau=\frac{\left[(5-3q)^{p+1}+(6q-5)^{p+1}\right](1-q)}{6^{p+1}q\left(1-q^{p+1}\right)}.$$

The integral boundaries that cause all these errors are chosen independently of q

Now, let us show the following Theorem 1 in [21] is not correct. For this, we give an example.

Theorem 2 *Suppose that* $\chi:[\rho,\sigma]\to\mathbb{R}$ *is a q - differentiable function on (ρ, σ) and 0 < q < 1. If* $\left|{}_{\rho}D_q\chi\right|$ *is convex and integrable function on [ρ, σ], then we possess the inequality*

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$$\frac{1}{6}\left|\chi(\rho)+4\chi\left(\frac{\rho+\sigma}{2}\right)+\chi(\sigma)-\frac{1}{(\sigma-\rho)}\int_{\rho}^{\sigma}\chi(\tau)\,{}_{\rho}d_{q}\tau\right| \quad (1)$$

$$\leq\frac{(\sigma-\rho)}{12}\left\{\frac{2q^{2}+2q+1}{q^{3}+2q^{2}+2q+1}\left|{}_{\rho}D_{q}\chi(\sigma)\right|+\frac{1}{3}\frac{6q^{3}+4q^{2}+4q+1}{q^{3}+2q^{2}+2q+1}\left|{}_{\rho}D_{q}\chi(\rho)\right|\right\}.$$

Example 1 *Let's choose* $\chi(\tau)=1-\tau$ *on* $[0,1]$ *and* $x(\tau)$ *satisfies the conditions of Theorem 2. On the other hand,* $|{}_{\rho}D_{q}\chi|=|{}_{\rho}D_{q}(1-\tau)|=1$ *is convex and integrable on* $[0,1]$ *Then we have*

$$\frac{1}{6}\left|\chi(\rho)+4\chi\left(\frac{\rho+\sigma}{2}\right)+\chi(\sigma)-\frac{1}{(\sigma-\rho)}\int_{\rho}^{\sigma}\chi(\tau)\,{}_{\rho}d_{q}\tau\right|=\frac{3+2q}{6(1+q)}. \quad (2)$$

Also,

$$\frac{(\sigma-\rho)}{12}\left\{\frac{2q^{2}+2q+1}{q^{3}+2q^{2}+2q+1}\left|{}_{\rho}D_{q}\chi(\sigma)\right|+\frac{1}{3}\frac{6q^{3}+4q^{2}+4q+1}{q^{3}+2q^{2}+2q+1}\left|{}_{\rho}D_{q}\chi(\rho)\right|\right\} \quad (3)$$

$$=\frac{1}{18}\frac{3q^{3}+5q^{2}+5q+2}{q^{3}+2q^{2}+2q+1}.$$

As we have seen, from (2) and (3) and for $q = (0, 1)$ we write

$$\frac{3+2q}{6(1+q)}\not\leq\frac{1}{18}\frac{3q^{3}+5q^{2}+5q+2}{q^{3}+2q^{2}+2q+1}.$$

For instance, choosing $q=\frac{1}{2}$ we have

$$\frac{4}{9}\not\leq\frac{7}{54}.$$

Therefore, Inequality (1) is not correct.

Similarly, other theorems can be shown to be false.

On the other hand, in [16], Alp, et al. showed that

$$\chi\left(\frac{\rho+\sigma}{2}\right)\not\leq\frac{1}{\sigma-\rho}\int_{\rho}^{\sigma}\chi(\varkappa)\,{}_{\rho}d_{q}\varkappa$$

and proved the following true q -Hermite-Hadamard inequalities convex functions on quantum integral:

Theorem 3 *If* $\chi:[\rho,\sigma]\rightarrow\mathbb{R}$ *be a convex differentiable function on* $[\rho,\sigma]$ *and and* $0<q<1$*. Then,* q *-Hermite-Hadamard inequalities*

$$\chi\left(\frac{q\rho+\sigma}{1+q}\right)\leq\frac{1}{\sigma-\rho}\int_{\rho}^{\sigma}\chi(\varkappa)\,{}_{\rho}d_{q}\varkappa\leq\frac{q\chi(\rho)+\chi(\sigma)}{1+q}.$$

As can be seen, q -Hermite-Hadamard inequality includes the value of $\chi\left(\frac{q\rho+\sigma}{1+q}\right)$. For this, Simpson's type quantum integral inequalities must also contain the value $\chi\left(\frac{q\rho+\sigma}{1+q}\right)$.

In this paper, motivated by the above results, by choosing the appropriate integral bounds we establish correct Simpson-type quantum integral inequalities obtained by using quantum integrals for functions whose absolute value of the q -derivatives are convex. Later, similar results for mappings whose powers of the absolute value of q -derivatives are convex are obtained via Hölder's inequality. Also, relations between special cases of these results and inequalities presented in the earlier works are examined.

Preliminaries and Definitions of q-Calculus

Throughout this paper, let $\rho<\sigma$ and $0<q<1$ be a constant. The following definitions and theorems for q - derivative and q- integral of a function x on $[\rho,\sigma]$ are given in [15,21].

Definition 1 [15,21]. *For a continuous function* $\chi:[\rho,\sigma]\rightarrow\mathbb{R}$ *then q - derivative of x at* $\varkappa\in[\rho,\sigma]$ *is characterized by the expression*

$${}_{\rho}D_{q}\chi(\varkappa)=\frac{\chi(\varkappa)-\chi(q\varkappa+(1-q)\rho)}{(1-q)(\varkappa-\rho)},\varkappa\neq\rho. \quad (4)$$

Since $\chi:[\rho,\sigma]\rightarrow\mathbb{R}$ is a continuous function, thus we have ${}_{\rho}D_{q}\chi(\rho)=\lim_{\varkappa\to\rho}{}_{\rho}D_{q}\chi(\varkappa)$. The function x is said to be q -differentiable on $[\rho,\sigma]$ if ${}_{\rho}D_{q}\chi(\tau)$ exists for all $\varkappa\in[\rho,\sigma]$. If $\rho=0$ in (4), then ${}_{0}D_{q}\chi(\varkappa)=D_{q}\chi(\varkappa)$, where $D_{q}\chi(\varkappa)$ is familiar q -derivative of x at $\varkappa\in[\rho,\sigma]$ defined by the expression ([12,23])

$$D_{q}\chi(\varkappa)=\frac{\chi(\varkappa)-\chi(q\varkappa)}{(1-q)\varkappa},\varkappa\neq0. \quad (5)$$

Definition 2 *[21, 22]. Let* $\chi:[\rho,\sigma]\rightarrow\mathbb{R}$ *be a continuous function. Then the q -definite integral on* $[\rho,\sigma]$ *is delineated as*

$$\int_{\rho}^{\varkappa}\chi(\tau)\,{}_{\rho}d_{q}\tau=(1-q)(\varkappa-\rho)\sum_{n=0}^{\infty}q^{n}\chi\left(q^{n}\varkappa+(1-q^{n})\rho\right) \quad (6)$$

for $\varkappa\in[\rho,\sigma]$.

If $\rho=0$ in (6), then $\int_{0}^{\varkappa}\chi(\tau)\,{}_{0}d_{q}\tau=\int_{0}^{\varkappa}\chi(\tau)d_{q}\tau$, where $\int_{0}^{\varkappa}\chi(\tau)d_{q}\tau$ is familiar q -definite integral on $[0,\varkappa]$ defined by the expression (see [14])

$$\int_{0}^{\varkappa}\chi(\tau)\,{}_{0}d_{q}\tau=\int_{0}^{\varkappa}\chi(\tau)d_{q}\tau=(1-q)\varkappa\sum_{n=0}^{\infty}q^{n}\chi(q^{n}\varkappa). \quad (7)$$

If $c\in(\rho,\varkappa)$, then the q -definite integral on $[c,\varkappa]$ is expressed as

$$\int_{c}^{\varkappa}\chi(\tau)\,{}_{\rho}d_{q}\tau=\int_{\rho}^{\varkappa}\chi(\tau)\,{}_{\rho}d_{q}\tau-\int_{\rho}^{c}\chi(\tau)\,{}_{\rho}d_{q}\tau. \quad (8)$$

$[n]_{q}$ notation

$$[n]_{q}=\frac{q^{n}-1}{q-1}$$

Lemma 1 [21] *For* $\alpha\in\mathbb{R}\setminus\{-1\}$*, the following formula holds:*

$$\int_{\rho}^{\varkappa}(\tau-\rho)^{\alpha}\,{}_{\rho}d_{q}\tau=\frac{(\varkappa-\rho)^{\alpha+1}}{[\alpha+1]_{q}}. \quad (9)$$

Main results

For convenience, we begin with some notations which will

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be used in what follows.

$$A_1(q)=2\frac{q[3]_q[6]_q-q^2}{[2]_q[3]_q[6]_q^3}+\frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q}-\frac{q^2+2q}{[6]_q}\right), \tag{10}$$

$$B_1(q)=\frac{2q^2[2]_q^2+[6]_q^2\left([6]_q-[3]_q\right)}{[2]_q^3[3]_q[6]_q^3}, \tag{11}$$

$$A_2(q)=2\frac{q[5]_q^2[6]_q[3]_q-q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{q^2}{[2]_q[3]_q}-\frac{q[5]_q}{[2]_q[6]_q} \tag{12}$$

$$-\frac{1}{[2]_q^3}\left[\frac{\left(q^2+2q\right)[5]_q}{[6]_q}-\frac{q+q^2}{[3]_q}\right]$$

and

$$B_2(q)=\frac{2q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{[6]_q(1+[2]_q^3)-[3]_q[5]_q(1+[2]_q^2)}{[2]_q^3[3]_q[6]_q}. \tag{13}$$

In order to easily prove our main results we first give a new identity including quantum integrals in the following.

Lemma 2 *Let $\chi:[\rho,\sigma]\to\mathbb{R}$ be a q- differentiable function on (ρ, σ) and 0 < q < 1. If ${}_\rho D_q\chi$ is continuous and integrable on [ρ, σ], then we possess the identity*

$$\frac{q\chi(\rho)+q^2[4]_q\chi\left(\frac{q\rho+\sigma}{1+q}\right)+\chi(\sigma)}{[6]_q}-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)\,{}_\rho d_q\tau \tag{14}$$

$$=q(\sigma-\rho)\int_0^1\psi(\tau)\,{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\,{}_0d_q\tau$$

where

$$\psi(\tau)=\begin{cases}\tau-\frac{1}{[6]_q} & \tau\in\left[0,\frac{1}{1+q}\right)\\ \tau-\frac{[5]_q}{[6]_q} & \tau\in\left[\frac{1}{1+q},1\right].\end{cases}$$

Proof. From the basic properties of quantum integral and the definition of $\psi(\tau)$, it follows that

$$\int_0^1\psi(\tau)\,{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\,{}_0d_q\tau$$

$$=\frac{[5]_q-1}{[6]_q}\int_0^{\frac{1}{1+q}}{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\,{}_0d_q\tau$$

$$+\int_0^1\left(\tau-\frac{[5]_q}{[6]_q}\right){}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\,{}_0d_q\tau.$$

Also, using the definition of q -derivative, we find that

$$\int_0^1\psi(\tau)\,{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\,{}_0d_q\tau \tag{15}$$

$$=\frac{[5]_q-1}{[6]_q}\int_0^{\frac{1}{1+q}}\frac{\chi(\tau\sigma+(1-\tau)\rho)-\chi(q\tau\sigma+(1-q\tau)\rho)}{\tau(1-q)(\sigma-\rho)}\,{}_0d_q\tau$$

$$+\int_0^1\left(\tau-\frac{[5]_q}{[6]_q}\right)\frac{\chi(\tau\sigma+(1-\tau)\rho)-\chi(q\tau\sigma+(1-q\tau)\rho)}{\tau(1-q)(\sigma-\rho)}\,{}_0d_q\tau.$$

Now, if we calculate the first quantum integral on the right side of the above equality by considering the definition 2, then we obtain

$$\int_0^{\frac{1}{1+q}}\frac{\chi(\tau\sigma+(1-\tau)\rho)-\chi(q\tau\sigma+(1-q\tau)\rho)}{\tau(1-q)(\sigma-\rho)}\,{}_0d_q\tau \tag{16}$$

$$=\frac{1}{(\sigma-\rho)}\frac{1}{1+q}\left\{\sum_{n=0}^{\infty}q^n\frac{\chi\left(\frac{q^n}{1+q}\sigma+\left(1-\frac{q^n}{1+q}\right)\rho\right)}{\frac{q^n}{1+q}}\right.$$

$$\left.-\sum_{n=0}^{\infty}q^n\frac{\chi\left(\frac{q^{n+1}}{1+q}\sigma+\left(1-\frac{q^{n+1}}{1+q}\right)\rho\right)}{\frac{q^n}{1+q}}\right\}$$

$$=\frac{1}{(\sigma-\rho)}\left\{\chi\left(\frac{q\rho+\sigma}{1+q}\right)-\chi(\rho)\right\}.$$

Similarly, we have

$$\int_0^1\left(\tau-\frac{[5]_q}{[6]_q}\right)\frac{\chi(\tau\sigma+(1-\tau)\rho)-\chi(q\tau\sigma+(1-q\tau)\rho)}{\tau(1-q)(\sigma-\rho)}\,{}_0d_q\tau \tag{17}$$

$$=\frac{1}{(\sigma-\rho)}\left\{\left(\frac{q-1}{q}\right)\sum_{n=0}^{\infty}q^n\chi\left(q^n\sigma+(1-q^n)\rho\right)+\frac{1}{q}\chi(\sigma)\right\}$$

$$-\frac{[5]_q}{[6]_q(\sigma-\rho)}\{\chi(\sigma)-\chi(\rho)\}$$

$$=\frac{1}{q(\sigma-\rho)}\chi(\sigma)-\frac{[5]_q}{[6]_q(\sigma-\rho)}\{\chi(\sigma)-\chi(\rho)\}-\frac{1}{q(\sigma-\rho)^2}\int_\rho^\sigma\chi(\tau)\,{}_\rho d_q\tau.$$

Multiplying the resulting equality by $q(\sigma-\rho)$ after substituting the identities (16) and (17) in (15), the desired result can be readily attained.

Corollary 4 *Under the assumptions of Lemma 2 with $q\to 1$, one has*

$$\frac{1}{6}\left[\chi(\rho)+4\chi\left(\frac{\rho+\sigma}{2}\right)+\chi(\sigma)\right]-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)d\tau$$

$$=q(\sigma-\rho)\int_0^1\psi(\tau)\chi'(\tau\sigma+(1-\tau)\rho)d\tau$$

which was presented by Alomari, et al. in [3]. Here, $\psi(\tau)$ is defined by

$$\psi(\tau)=\begin{cases}\tau-\frac{1}{6} & \tau\in\left[0,\frac{1}{2}\right)\\ \tau-\frac{5}{6} & \tau\in\left[\frac{1}{2},1\right].\end{cases}$$

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Now, we examine how the results come out when we use a function whose quantum derivatives in modulus are convex.

Theorem 5 *Suppose that* $\chi:[\rho,\sigma]\to\mathbb{R}$ *is a q - differentiable function on* (ρ,σ) *and* $0<q<1$*. If* $\left|{}_\rho D_q\chi\right|$ *is convex and integrable function on* $[\rho,\sigma]$*, then we possess the inequality*

$$\left|\frac{q\chi(\rho)+q^2[4]_q\chi\left(\frac{q\rho+\sigma}{1+q}\right)+\chi(\sigma)}{[6]_q}-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)\,{}_\rho d_q\tau\right| \qquad (18)$$

$$\le q(\sigma-\rho)$$

$$\times\left\{\left|{}_\rho D_q\chi(\rho)\right|\left[A_1(q)+A_2(q)\right]+\left|{}_\rho D_q\chi(\sigma)\right|\left[B_1(q)+B_2(q)\right]\right\}$$

where $A_1(q)$, $A_2(q)$, $B_1(q)$ and $B_2(q)$ are defined as in (10)-(13), respectively.

Proof. If we take the absolute value of both sides of (14), then we have

$$\left|\frac{q\chi(\rho)+q^2[4]_q\chi\left(\frac{q\rho+\sigma}{1+q}\right)+\chi(\sigma)}{[6]_q}-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)\,{}_\rho d_q\tau\right| \qquad (19)$$

$$\le q(\sigma-\rho)\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau$$

$$+q(\sigma-\rho)\int_{\frac{1}{1+q}}^{1}\left|\tau-\frac{[5]_q}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau.$$

For the first expression on the right side of the inequality (19), seeing that $\left|{}_\rho D_q\chi(\tau)\right|$ is convex on $[\rho,\sigma]$, it follows that

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau$$

$$\le\left|{}_\rho D_q\chi(\rho)\right|\int_0^{\frac{1}{1+q}}(1-\tau)\left|\tau-\frac{1}{[6]_q}\right|{}_0d_q\tau+\left|{}_\rho D_q\chi(\sigma)\right|\int_0^{\frac{1}{1+q}}\tau\left|\tau-\frac{1}{[6]_q}\right|{}_0d_q\tau.$$

Now, calculating the quantum integrals on the right side of the above inequality by considering the case when $\rho=0$ of Lemma 1, we find that

$$\int_0^{\frac{1}{1+q}}(1-\tau)\left|\tau-\frac{1}{[6]_q}\right|{}_0d_q\tau$$

$$=2\int_0^{\frac{1}{[6]_q}}\tau\left(\frac{1}{[6]_q}-\tau\right){}_0d_q\tau+\int_0^{\frac{1}{1+q}}\tau\left(\tau-\frac{1}{[6]_q}\right){}_0d_q\tau$$

$$=2\frac{q[3]_q[6]_q-q^2}{[2]_q[3]_q[6]_q^3}+\frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q}-\frac{q^2+2q}{[6]_q}\right),$$

and

$$\int_0^{\frac{1}{1+q}}\tau\left|\tau-\frac{1}{[6]_q}\right|{}_0d_q\tau$$

$$=2\int_0^{\frac{1}{[6]_q}}\tau\left(\frac{1}{[6]_q}-\tau\right){}_0d_q\tau+\int_0^{\frac{1}{1+q}}\tau\left(\tau-\frac{1}{[6]_q}\right){}_0d_q\tau$$

$$=\frac{2q^2[2]_q^2+[6]_q^2\left([6]_q-[3]_q\right)}{[2]_q^3[3]_q[6]_q^3}.$$

Then, one has the result

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau \qquad (20)$$

$$\le\left|{}_\rho D_q\chi(\rho)\right|\left[2\frac{q[3]_q[6]_q-q^2}{[2]_q[3]_q[6]_q^3}+\frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q}-\frac{q^2+2q}{[6]_q}\right)\right]$$

$$+\left|{}_\rho D_q\chi(\sigma)\right|\frac{2q^2[2]_q^2+[6]_q^2\left([6]_q-[3]_q\right)}{[2]_q^3[3]_q[6]_q^3}.$$

If similar operations are applied for the other expression in (19), due to the convexity of $\left|{}_\rho D_q\chi(\sigma)\right|$, then one possesses the inequality

$$\int_{\frac{1}{1+q}}^{1}\left|\tau-\frac{[5]_q}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau \qquad (21)$$

$$\le\left|{}_\rho D_q\chi(\rho)\right|\left\{2\frac{q[5]_q^2[6]_q[3]_q-q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{q^2}{[2]_q[3]_q}-\frac{q[5]_q}{[2]_q[6]_q}\right.$$

$$\left.-\frac{1}{[2]_q^3}\left[\frac{(q^2+2q)[5]_q}{[6]_q}-\frac{q+q^2}{[3]_q}\right]\right\}$$

$$+\left|{}_\rho D_q\chi(\sigma)\right|\left\{\frac{2q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{[6]_q(1+[2]_q^3)-[3]_q[5]_q(1+[2]_q^2)}{[2]_q^3[3]_q[6]_q}\right\}.$$

Substituting the inequalities (20) and (21) in (19), the desired result can be readily attained. Hence, the proof is completed.

Corollary 6 *If we take the limit of both sides of (18) as* $q\to1$*, then the inequality (18) yields the result*

$$\left|\frac{1}{6}\left[\chi(\rho)+4\chi\left(\frac{\rho+\sigma}{2}\right)+\chi(\sigma)\right]-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)d\tau\right|$$

$$\le\frac{5(\sigma-\rho)}{72}\left[\left|\chi'(\rho)\right|+\left|\chi'(\sigma)\right|\right]$$

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which is Simpson type inequality for functions whose absolute values of derivatives are convex. This result was provided by Alomari, et al. in [1].

We observe how the inequalities come out when we use mappings whose q -derivatives in modulus at certain powers are convex.

Theorem 7 *Assume that* $\chi:[\rho,\sigma]\to\mathbb{R}$ *is a* q *-differentiable function on* (ρ, σ) *and* $0 < q < 1$. *If* $|_{\rho} D_q\chi|^s$ *is convex and integrable function on* $[\rho, \sigma]$ *where* $s > 1$ *with* $\frac{1}{r}+\frac{1}{s}=1$, *then one has the result*

$$\left|\frac{q\chi(\rho)+q^2[4]_q\chi\left(\frac{q\rho+\sigma}{1+q}\right)+\chi(\sigma)}{[6]_q}-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)\,{}_\rho d_q\tau\right|$$

$$\leq q(\sigma-\rho)$$

$$\times\left\{\left(\frac{q^{2r}[4]_q^r}{[2]_q^{r+1}[6]_q^r}\right)^{\frac{1}{r}}\left(\frac{q^2+2q}{[2]_q^3}|{}_\rho D_q\chi(\rho)|^s+\frac{1}{[2]_q^3}|{}_\rho D_q\chi(\sigma)|^s\right)^{\frac{1}{s}}\right.$$

$$+\left(\frac{[2]_q^{r+1}[5]_q^r-q^r[4]_q^r}{[2]_q^{r+1}[6]_q^r}\right)^{\frac{1}{r}}$$

$$\left.\times\left(\frac{q^3+q^2-q}{[2]_q^3}|{}_\rho D_q\chi(\rho)|^s+\frac{q^2+2q}{[2]_q^3}|{}_\rho D_q\chi(\sigma)|^s\right)^{\frac{1}{s}}\right\}.$$

Proof. We reconsider the inequality (19). Applying Hölder's inequality to the first integral on the right side of (19), due to the convexity of $|_{\rho} D_q\chi|^s$, it is found that

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau$$

$$\leq\left(\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|^r{}_0d_q\tau\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{1+q}}\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|^s{}_0d_q\tau\right)^{\frac{1}{s}}$$

$$\leq\left(\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|^r{}_0d_q\tau\right)^{\frac{1}{r}}\left(\frac{q^2+2q}{(1+q)^3}|{}_\rho D_q\chi(\rho)|^s+\frac{1}{(1+q)^3}|{}_\rho D_q\chi(\sigma)|^s\right)^{\frac{1}{s}}.$$

We also observe that

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|^r{}_0d_q\tau=\frac{(1-q)}{1+q}\sum_{n=0}^{\infty}q^n\left|\frac{q^n}{1+q}-\frac{1}{[6]_q}\right|^r$$

$$\leq\frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}q^n\left|\frac{1}{[2]_q}-\frac{1}{[6]_q}\right|^r$$

$$=\left(\frac{1}{[2]_q}-\frac{1}{[6]_q}\right)^r\frac{(1-q)}{[2]_q}\frac{1}{1-q}$$

$$=\frac{q^{2r}[4]_q^r}{[2]_q^{r+1}[6]_q^r}.$$

Similarly, using Hölder's inequality for the second integral on the right side of (19), owing to the convexity of $|_{\rho} D_q\chi|^s$, we find that

$$\int_{\frac{1}{1+q}}^{1}\left|\tau-\frac{[5]_q}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau$$

$$\leq\left(\int_{\frac{1}{1+q}}^{1}\left|\tau-\frac{[5]_q}{[6]_q}\right|^r{}_0d_q\tau\right)^{\frac{1}{r}}$$

$$\times\left(|{}_\rho D_q\chi(\rho)|^s\frac{q^3+q^2-q}{(1+q)^3}+|{}_\rho D_q\chi(\sigma)|^s\frac{q^2+2q}{(1+q)^3}\right)^{\frac{1}{s}}.$$

We also have

$$\int_{\frac{1}{[2]_q}}^{1}\left|\tau-\frac{[5]_q}{[6]_q}\right|^r{}_0d_q\tau=\int_0^1\left|\tau-\frac{[5]_q}{[6]_q}\right|^r{}_0d_q\tau-\int_0^{\frac{1}{[2]_q}}\left|\tau-\frac{[5]_q}{[6]_q}\right|^r{}_0d_q\tau$$

$$\leq\frac{[2]_q^{r+1}[5]_q^r-q^r[4]_q^r}{[2]_q^{r+1}[6]_q^r}.$$

Finally, if we substitute the above results in (19), then we obtain the desired inequality.

Theorem 8 *Supposing that* $\chi:[\rho,\sigma]\to\mathbb{R}$ *is a* q *- differentiable function on* (ρ, σ) *and* $0 < q < 1$. *If* $|_{\rho} D_q\chi|^s$ *is convex and integrable function on* $[\rho, \sigma]$ *where* s 1, *then one has the result*

$$\frac{q\chi(\rho)+q^2[4]_q\chi\left(\frac{q\rho+\sigma}{1+q}\right)+\chi(\sigma)}{[6]_q}-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)\,{}_\rho d_q\tau \tag{22}$$

$$\leq q(\sigma-\rho)\left\{\left(\frac{2q}{[2]_q[6]_q^2}+\frac{q^3[3]_q-q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{s}}\right.$$

$$\times\left[A_1(q)|{}_\rho D_q\chi(\rho)|^s+B_1(q)|{}_\rho D_q\chi(\sigma)|^s\right]^{\frac{1}{s}}$$

$$+\left(2q\frac{[5]_q^2}{[2]_q[6]_q^2}+\frac{1}{[2]_q}-\frac{[5]_q}{[6]_q}-\frac{[5]_q[2]_q^2-[6]_q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{s}}$$

$$\left.\times\left[A_2(q)|{}_\rho D_q\chi(\rho)|^s+B_2(q)|{}_\rho D_q\chi(\sigma)|^s\right]^{\frac{1}{s}}\right\}$$

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where $A_1(q)$, $A_2(q)$, $B_1(q)$ and $B_2(q)$ are defined as in (10)-(13), respectively.

Proof. We consider the inequality (19). Applying power mean inequality to the first integral on the right side of (19), we find that

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|^{1-\frac{1}{s}}\left|\tau-\frac{1}{[6]_q}\right|^{\frac{1}{s}}\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau$$

$$\leq\left(\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|{}_0d_q\tau\right)^{1-\frac{1}{s}}$$

$$\times\left(\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|^s{}_0d_q\tau\right)^{\frac{1}{s}}.$$

The operations which have been used in the proof of theorem 5 are applied by considering that $\left|{}_\rho D_q\chi(\tau)\right|^s$ is convex on [ρ, σ] it is easy to see that

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|^s{}_0d_q\tau$$

$$\leq\left|{}_\rho D_q\chi(\rho)\right|^s\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|(1-\tau){}_0d_q\tau+\left|{}_\rho D_q\chi(\sigma)\right|^s\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\tau\,{}_0d_q\tau$$

$$=\left|{}_\rho D_q\chi(\rho)\right|^sA_1(q)+\left|{}_\rho D_q\chi(\sigma)\right|^sB_1(q),$$

where $A_1(q)$ and $B_1(q)$ are defined as in (10) and (11), respectively. Also, from the definition of quantum integral, we observe that

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|{}_0d_q\tau$$

$$=2\int_0^{\frac{1}{[6]_q}}\left(\frac{1}{[6]_q}-\tau\right){}_0d_q\tau+\int_0^{\frac{1}{1+q}}\left(\tau-\frac{1}{[6]_q}\right){}_0d_q\tau$$

$$=\frac{2q}{[2]_q[6]_q^2}+\frac{q^3[3]_q-q}{[6]_q(1+q)^3}.$$

Thus, we obtain the inequality

$$\int_0^{\frac{1}{1+q}}\left|\tau-\frac{1}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau \tag{23}$$

$$\leq\left(\frac{2q}{[2]_q[6]_q^2}+\frac{q^3[3]_q-q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{s}}$$

$$\times\left[\left|{}_\rho D_q\chi(\rho)\right|^sA_1(q)+\left|{}_\rho D_q\chi(\sigma)\right|^sB_1(q)\right]^{\frac{1}{s}}.$$

Similarly, for the second integral on the right side of (19), we have

$$\int_{\frac{1}{1+q}}^1\left|\tau-\frac{[5]_q}{[6]_q}\right|\left|{}_\rho D_q\chi(\tau\sigma+(1-\tau)\rho)\right|{}_0d_q\tau \tag{24}$$

$$\leq\left(2q\frac{[5]_q^2}{[2]_q[6]_q^2}+\frac{1}{[2]_q}-\frac{[5]_q}{[6]_q}-\frac{[5]_q[2]_q^2-[6]_q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{s}}$$

$$\left[\left|{}_\rho D_q\chi(\rho)\right|^sA_2(q)+\left|{}_\rho D_q\chi(\sigma)\right|^sB_2(q)\right]^{\frac{1}{s}}.$$

Should we substitute the inequalities (23) and (24) in (19), and we then capture the desired result which finishes the proof.

Corollary 9 *If we take the limit of both sides of (22) as* $q\to 1$, *then the inequality (22) reduces to the result*

$$\left|\frac{1}{6}\left[\chi(\rho)+4\chi\left(\frac{\rho+\sigma}{2}\right)+\chi(\sigma)\right]-\frac{1}{(\sigma-\rho)}\int_\rho^\sigma\chi(\tau)d\tau\right|$$

$$\leq\frac{1}{(1296)^{\frac{1}{s}}}\left(\frac{5}{72}\right)^{1-\frac{1}{s}}(\sigma-\rho)$$

$$\times\left\{\left[61|\chi'(\rho)|^s+29|\chi'(\sigma)|^s\right]^{\frac{1}{s}}+\left[29|\chi'(\rho)|^s+61|\chi'(\sigma)|^s\right]^{\frac{1}{s}}\right\}$$

which was presented by Alomari, et al. in [3].

Conclusion

In this work, Simpson-type quantum integral inequalities found to be incorrect by Tunç were corrected. New and correct quantum integral inequalities were thus developed by using mappings whose absolutes value of q -derivatives are convex. Also, relations between special cases of these results and inequalities given in the earlier works are observed. Also, this paper describes how to find quantum integral inequalities for convex functions.

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Author contributions

The authors declare that the study was realized in collaboration with the same responsibility.

Availability of data and material

Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current research.

References

1. Alomari MW, Darus M, Dragomir SS. New inequalities of Simpson's type for s-convex functions with applications. RGMIA Res Rep Coll. 2009;12(4)9. Available from: https://rgmia.org/papers/v12n4/Simpson.pdf
2. Dragomir SS, Agarwal RP, Cerone P. On Simpson's inequality and applications. J Inequal Appl. 2000;5:533-579. Available from: https://rgmia.org/papers/v2n3/Simpsurv.pdf
3. Sarikaya MZ, Set E, Özdemir ME. On new inequalities of Simpson's type for convex functions. RGMIA Res Rep Coll. 2010;13(2)2. Available from: https://doi.org/10.1016/j.camwa.2010.07.033
4. Sarikaya MZ, Budak H, Erden S. On new inequalities of Simpson's type for generalized convex functions. Korean J Math. 2019;27(2):279-295. Available from: http://dx.doi.org/10.11568/kjm.2019.27.2.279
5. Liu Z. An inequality of Simpson type. Proc Royal Soc A Math Phys Eng Sci. 2005;461(2059):2155-2158. Available from: https://www.sciepub.com/reference/162832
6. Erden S, Iftikhar S, Delavar RM, Kumam P, Thounthong P, Kumam W. On generalizations of some inequalities for convex functions via quantum integrals. Rev Real Acad Cienc Exactas Fisicas Naturales. 2020;114(3):1–15. Available from: https://link.springer.com/article/10.1007/s13398-020-00841-3
7. Ernst T. A comprehensive treatment of -calculus. Basel: Springer; 2016.
8. Ernst T. A method for -calculus. J Nonlinear Math Phys. 2003;10(4):487-525.
9. Ernst T. The history of -calculus and new method. Uppsala: Department of Mathematics, Uppsala University; 2000.
10. Gauchman H. Integral inequalities in -calculus. Comput Math Appl. 2004;47:281-300.
11. Jackson FH. On a -definite integrals. Quart J Pure Appl Math. 1910;41:193-203.
12. Kac V, Cheung P. Quantum calculus. New York: Springer; 2002.
13. Agarwal R. A propos d'une note de M. Pierre Humbert. Reports from the Academy of Sciences. 1953;236(21):2031-2032.
14. Al-Salam W. Some fractional q-integrals and q-derivatives. Proc Edinburgh Math Soc. 1966/1967;15(2):135–40. Available from: https://doi.org/10.1017/S0013091500011469
15. Tariboon J, Ntouyas SK. Quantum calculus on finite intervals and applications to impulsive difference equations. Adv Difference Equ. 2013;2013:282:19. Available from: http://dx.doi.org/10.1186/1687-1847-2013-282
16. Alp N, Sarikaya MZ, Kunt M, Iscan I. Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. J King Saud Univ Sci. 2018;30:193-203.
17. Noor MA, Noor KI, Awan MU. Some quantum estimates for Hermite-Hadamard inequalities. Appl Math Comput. 2015;251:675-679. Available from: http://dx.doi.org/10.1016/j.amc.2014.11.090
18. Noor MA, Noor KI, Awan MU. Some quantum integral inequalities via preinvex functions. Appl Math Comput. 2015;269:242-251. Available from: http://dx.doi.org/10.1016/j.amc.2015.07.078
19. Noor MA, Noor KI, Awan MU. Quantum Ostrowski inequalities for -differentiable convex functions. J Math Inequal. 2016;10:1013-1018.
20. Budak H, Erden S, Ali MA. Simpson and Newton-type inequalities for convex functions via newly defined quantum integrals. Math Methods Appl Sci. 2021;44(1):378-390. https://doi.org/10.1002/mma.6742
21. Tunç M, Göv E, Balgeçti S. Simpson type quantum integral inequalities for convex functions. Miskolc Math Notes. 2018;19:649-664. Available from: http://dx.doi.org/10.18514/MMN.2018.1661
22. Tariboon J, Ntouyas SK. Quantum integral inequalities on finite intervals. J Inequal Appl. 2014;2014:121:13. Available from: https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/1029-242X-2014-121
23. Jackson FH. -Difference equations. Amer J Math. 1910;32(4):305-314.

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